A STOCHASTIC MODEL OF THE LONG-RANGE FINANCIAL STATUS OF THE OASDI PROGRAM

ACTUARIAL STUDY NO. 128

K. Mark Bye, ASA
Michael L. Miller
Michael Morris, ASA
Sven H. Sinclair, FSA, MAAA

Social Security Administration
Office of the Chief Actuary

March 2024
SSA Pub. No. 11-11555

FOREWORD

The purpose of the OCACT Stochastic Model (OSM) is to provide probability distributions for the range of possible future experience of the OASDI program under current law. This probabilistic representation of uncertainty augments the presentations of low-cost and high-cost alternative scenarios and sensitivity analyses that have traditionally been included in the Annual Reports of the OASDI Board of Trustees (Trustees Reports).

The OSM was introduced in the 2003 Trustees Report. Actuarial Study No. 117, published in September 2004, provided details about the model's first stage of development and the projections produced using it. The OSM was enhanced in the 2021 Trustees Report to include parameter uncertainty. This study (Actuarial Study No. 128) provides details about the version of the OSM used for the 2023 Trustees Report and the projections produced using it. We expect to continue making improvements and refinements to the OSM in the future.

Many of our colleagues, both inside and outside the Office of the Chief Actuary, provided input to the development of this study. We would like to acknowledge Anthony Cheng, David Pattison, Jason Schultz, and Robert Weathers for their technical expertise and support in developing the enhanced parameter uncertainty methodology. Kyle Burkhalter, Sharon Chu, Skylar Lowe, Johanna Maleh, Kent Morgan, Daniel Nickerson, Karen Rose, and Katie Sutton contributed to the ongoing development of the model and the writing and editing of this study. Erica Ciccotto assisted in the publication of this study.

Karen P. Glenn, FSA, EA, MAAA
Deputy Chief Actuary

Stephen C. Goss, ASA, MAAA
Chief Actuary

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A STOCHASTIC MODEL[1] OF THE LONG-RANGE FINANCIAL STATUS OF THE OASDI PROGRAM

I. INTRODUCTION

The Annual Report of the Board of Trustees of the Federal Old-Age and Survivors Insurance (OASI) and Disability Insurance (DI) Trust Funds includes three separate sets of long-range (75-year) assumptions for key demographic, economic, and programmatic variables that affect the future financial status of the combined OASI and DI (OASDI) programs. The intermediate (alternative II) set of assumptions represents the Trustees' best estimate for future experience, while the low-cost (alternative I) and high-cost (alternative III) sets of assumptions are more and less favorable, respectively, from the perspective of the trust funds' future financial outlook. The Office of the Chief Actuary (OCACT) of the Social Security Administration (SSA) uses the three sets of assumptions to project the principal factors affecting the financial status of the OASDI program. Taken together, these three projections give policymakers a sense of the range of variation in the assumptions and in the financial status of the OASDI program. However, this deterministic approach makes no attempt to assign values to the likelihood of these sets of assumptions.

Stochastic variation is introduced by developing equations based on standard time-series models. Generally, an equation may include the following: the variable's prior-period values, prior-period error terms, and other variables. In addition, each equation includes a random error term. The data used for the regressions depend on the nature and quality of the historical data. Projected values for each variable in each year are computed using Monte Carlo techniques to assign the degree of stochastic variation around the Trustees' intermediate assumptions. Each simulation projects annual values for each variable over the 75-year period, in addition to summary measures of the financial status of the combined OASI and DI Trust Funds (e.g., the long-range actuarial balance).

The OCACT Stochastic Model (OSM) assigns random variation for some of the key demographic, economic, and programmatic assumptions. These include total fertility rates, changes in mortality rates, new arrival lawful permanent resident (LPR) and other-than-LPR immigration levels, rates of adjustment of status (from other-than-LPR to LPR), rates of legal emigration (from the population of citizens and LPRs), changes in the Consumer Price Index, changes in average real wages, unemployment rates, trust fund real interest rates, and disability incidence and recovery rates. The OSM is designed so that the projected values for each variable are centered on the intermediate assumptions of the Trustees Report—at the time of this study, the 2023 Trustees

[1] A stochastic model is used for projecting a probability distribution of potential outcomes. Such models allow for random variation in one or more variables through time. The random variation is generally based on fluctuations observed in historical data for a selected period. A large number of simulations, each of which reflects random variation in the variable(s), produce a distribution of potential outcomes. This contrasts with a deterministic model, which has specified assumptions for, and relationships among, variables. Under such a model, any specified set of assumptions fully determines a single outcome directly reflecting the specifications. For other definitions of terms that are used in this study, see the 2023 Trustees Report at www.ssa.gov/oact/TR/2023/VI_I_glossary.html#1042403.

Report. This updated study[2] documents OSM version 2023.1. OSM version 2023.1 is the model referred to in *The 2023 Annual Report of the Board of Trustees of the Federal Old-Age and Survivors Insurance and Federal Disability Insurance Trust Funds*.[3] A stochastic model can incorporate parameter uncertainty for the expected means or for all parameters. Parameter uncertainty allows regression coefficients to vary between simulations. Parameter uncertainty is discussed in more detail in section II.B. Appendix E discusses why the OSM version 2023.1 uses parameter uncertainty for the expected mean rather than full parameter uncertainty.

The results presented here should be interpreted with caution and with a full understanding of the inherent limitations of the model. Results are sensitive to equation specifications, degrees of interdependence among variables, and the historical periods used for estimating model coefficients. For some variables, recent historical variation may not provide a realistic representation of the potential variation for the future. In addition, results would differ if variables other than those mentioned above (such as labor force participation rates, retirement rates, marriage rates, and divorce rates) were also allowed to vary randomly. Finally, if no parameter uncertainty or full parameter uncertainty were used instead of parameter uncertainty for the expected mean, then results would be different.

The remaining chapters of this study provide detailed information about the OSM. Chapter II presents the general approach to stochastic modeling as well as how parameter uncertainty was introduced into the model. Chapter III presents the equations used to model random variation in the assumptions. Chapter IV presents projection results, including the projected probability distributions for the stochastic assumptions and the summary actuarial measures used to assess the long-range financial status of the OASDI program. Results with and without parameter uncertainty are shown. As expected, results with parameter uncertainty are distributed in a wider range. Chapter IV also presents a sensitivity analysis for each stochastic assumption.

Chapter V includes eight appendices to this study. Appendix A explains the overall structure of the OSM and its modules.[4] Appendix B contains background material on various financial estimates of the OASDI program. Appendices C and D contain introductions to time-series modeling and Monte Carlo simulation, respectively. Appendix E explains why the Trustees only use parameter uncertainty for the expected mean rather than full parameter uncertainty. Appendix F provides additional details of the time-series equations used in the OSM as well as the projected central tendency parameters and bounding statistics. Appendix G is a bibliography of references used in the development of this study.

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[2] The first version of the study was based on OSM version 2004.1 and is located at www.ssa.gov/oact/NOTES/pdf_studies/study117.pdf.

[3] See www.ssa.gov/OACT/TR/2023/tr2023.pdf.

[4] A module is defined as a section of related program code.

II. STOCHASTIC MODEL

This chapter describes the OSM's general approach to introducing stochastic variation into some of the key demographic, economic, and programmatic assumptions. This variation is modeled using standard time-series techniques. Section A describes the basic stochastic model without parameter uncertainty for autoregressive moving average (ARMA) models, where the only source of random variation is from the error terms. Section B describes how parameter uncertainty is introduced into the basic model for ARMA models. Three of the economics variables use a vector autoregression model, which is described in detail in section III.D and appendix C.

A. BASIC MODEL

The equations used to model the assumptions are mostly ARMA models with the additional requirement that the mean of the variable Y_t is always equal to its value under the intermediate assumptions of the 2023 Trustees Report. The value of the variable under the 2023 Trustees Report intermediate assumptions is written Y_t^{TR}. In general, Y_t^{TR} changes in the early years of the projection period as it grades into an ultimate value that is attained in the later years. We call the resulting model a *modified autoregressive moving average model.*

An ARMA(p,q) model with p autoregressive lags and q moving average lags takes the form

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

If this time series is stationary, then the mean μ_ϕ of this equation is computed to be

$$\mu_\phi = \frac{\phi_0}{1 - \phi_1 - \phi_2 - \ldots - \phi_p}.$$

The above ARMA(p,q) equation can be written in "deviations form", with $y_t = Y_t - \mu_\phi$, such that

$$Y_t = \mu_\phi + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

For the modified ARMA model, there is the additional requirement that the mean of the variable Y_t is always equal to its value under the 2023 Trustees Report intermediate alternative assumptions. To satisfy this requirement, the above equation is re-written as

$$Y_t = Y_t^{TR} + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

In this basic model, variation in Y_t over the projection period would come from the correlations of the variable with itself over prior periods and the random shocks from the error term ε_t.

As the mean is fixed and equal to the value under the 2023 Trustees Report intermediate assumptions (i.e., Y_t^{TR}), the cumulative average of a time-series of simulated Y_t, over a sufficiently long enough time series, converges to its assumed mean, Y_t^{TR}. This result can be seen, to a certain degree, in the simulations without parameter uncertainty presented in Chapter IV. Forcing the mean equal to the values under the Trustees Report intermediate assumptions has the advantage of not allowing shocks, such as the COVID-19 pandemic, to affect stochastic projections from one Trustees Report to the next Trustees Report as much, other than increasing the variance of the projections.

B. INCORPORATING PARAMETER UNCERTAINTY INTO THE BASIC MODEL

In the previous section, the estimated parameters $\mu_\phi, \phi_1, \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q$ are assumed to be fixed for each equation and to not vary from simulation to simulation. However, these parameters are only estimates of the true parameters and are obtained from a sample of the true population. Thus, these estimates should not be treated as though they are certain.

Econometric software packages (including EViews, which is used for the OSM) produce not only the parameter estimates of the equation, but also the parameter variance-covariance matrix associated with the estimated equation.[5] When incorporating parameter uncertainty, this variance-covariance matrix **V**, which is $(1+p+q) \times (1+p+q)$ in dimension, is used to generate parameters for each simulation. This is done by multiplying the lower triangular matrix **L**, which is the Cholesky decomposition[6] of the variance-covariance matrix **V** (such that $\mathbf{V} = \mathbf{LL'}$), by a vector of independent standard normal random variables ($\vec{Z}$) to produce a vector of adjustments that are added to the estimated parameters. For an ARMA(p,q) equation, the process to obtain parameters adjusted for parameter uncertainty can be written in matrix notation in the following way:

$$\begin{bmatrix} \mu_\psi \\ \psi_1 \\ \dots \\ \psi_p \\ \varphi_1 \\ \dots \\ \varphi_q \end{bmatrix} = \begin{bmatrix} \mu_\phi \\ \phi_1 \\ \dots \\ \phi_p \\ \theta_1 \\ \dots \\ \theta_q \end{bmatrix} + \mathbf{L}\vec{Z}$$

The vectors above are $(1+p+q) \times 1$ in dimension. μ_ψ is the mean term adjusted for parameter uncertainty; $\psi_1, \dots, \psi_p$ are the autoregressive parameters adjusted for parameter uncertainty; $\varphi_1, \dots, \varphi_q$ are the moving average parameters adjusted for parameter uncertainty; $\mu_\phi, \phi_1, \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q$ are the parameters estimated originally from the ARMA(p,q) equation; and the product $\mathbf{L}\vec{Z}$ is the vector of adjustments that is added to the vector of originally estimated parameters.

[5] Variance-covariance matrices from these ARMA models are calculated using Newey-West estimators. Newey-West estimators extend Ordinary Least Square (OLS) estimators to overcome possible issues with serial correlation and heteroskedasticity.

[6] See appendix C for details on this technique.

Each simulation has its own vector of adjustments such that the same parameters adjusted for parameter uncertainty are used throughout the entire projection period, but the parameters vary from one simulation to the next.

Thus, for simulations with parameter uncertainty, there is a different mean associated with each simulation, and, assuming that the ARMA(*p,q*) process with parameters adjusted for parameter uncertainty is stationary, the mean implied by these parameters is $\hat{\mu}_{\psi}$. This mean is different from the process mean from the originally estimated equation, $\hat{\mu}_{\phi}$. The mean to be used in the stochastic simulation with parameter uncertainty is:

$$\mu_t^{PU} = \mu_t^{TR} + \zeta_t * (\hat{\mu}_{\psi} - \hat{\mu}_{\phi})$$

where μ_t^{TR} is the 2023 Trustees Report intermediate assumption value in year t, and ζ_t is the value of a 10-year phase-in factor in year t. The 10-year phase-in factor is applied to the difference between the $\hat{\mu}_{\psi}$ and $\hat{\mu}_{\phi}$ to avoid a sudden disconnect from the historical period to the beginning of the projection:

$$\zeta_t = min\left(1, \frac{t - (TRYEAR - 1)}{10}\right)$$

where TRYEAR is the Trustees Report year (2023).

The equation for performing simulations with parameter uncertainty becomes:

$$Y_t = \mu_t^{PU} + \psi_1(Y_{t-1} - \mu_{t-1}^{PU}) + \psi_2(Y_{t-2} - \mu_{t-2}^{PU}) + \ldots + \psi_p\left(Y_{t-p} - \mu_{t-p}^{PU}\right) + \varepsilon_t + \varphi_1\varepsilon_{t-1} + \varphi_2\varepsilon_{t-2} + \ldots + \varphi_q\varepsilon_{t-q}$$

where the ψ's and the φ's are the autoregressive and moving average parameters with parameter uncertainty, respectively. The $(Y_{t-n} - \mu_{t-n}^{PU})$ terms in the equation above are zero for all years $(t - n)$ prior to 2023. When running the model with parameter uncertainty for the estimated mean only, the autoregressive and moving average parameters calculated above are ignored and assumed to be the same values as without parameter uncertainty.

Some of the equations estimated under full parameter uncertainty can be nonstationary; that is, the variance in the projections grows exponentially over time. As a consequence, the range between the 2.5th and the 97.5th percentiles of the projections would grow over time. A non-stationary process is inconsistent with the assumptions under which the equations were estimated. Such nonstationary runs are eliminated from our simulations.

For an ARMA(*p,q*) process, the set of parameters will pass the stationarity test if the largest eigenvalue of the square matrix (*p*×*p*) lies inside the unit circle:

$$\begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_{p-1} & \psi_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

For parameter uncertainty for the estimated mean, a stationarity check is unnecessary since the original autoregressive parameters are unchanged from the original equations, which are known and required to be stationary. In other words, the $\mu_\psi, \phi_1, \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q$ parameters from above are used (and not the $\mu_\phi, \psi_1, \dots, \psi_p, \varphi_1, \dots, \varphi_q$ parameters). Since some equations have a high percentage of nonstationary draws and the mean parameter uncertainty is the most significant, the estimates used in the 2023 Trustees Report only allow for parameter uncertainty for the estimated mean term. For more details on the various forms of parameter uncertainty, see appendix E. In the rest of this study, unless otherwise specified, "parameter uncertainty" refers to the results of parameter uncertainty for the estimated mean.

A general approach to perform stochastic simulations with parameter uncertainty for an ARMA(*p,q*) model is described in this section. Pattison (2006) describes how stochastic simulations with and without parameter uncertainty could be implemented for the total fertility rate variable using an ARMA(4,1) equation, the same equation specification that is used in the OSM. However, Pattison takes a slightly different approach and does not estimate the means directly as we do for the OSM.

III. EQUATION SELECTION AND PARAMETER ESTIMATION

Equations were selected for a set of assumption variables that include total fertility rates, changes in mortality rates, new arrival lawful permanent resident (LPR) and other-than-LPR immigration levels, rates of adjustment of status (from other-than-LPR to LPR), rates of legal emigration (from the population of citizens and LPRs), changes in the Consumer Price Index, changes in average real wages, unemployment rates, trust fund real interest rates, and disability incidence and recovery rates. The parameters of the equations were estimated using standard time-series modeling techniques, and then modified so that the projected variation was centered on the 2023 Trustees Report intermediate assumptions. Appendix C discusses the theory behind time-series modeling. This chapter briefly describes and presents each equation, while appendix F provides more detailed information and statistics.

For most variables, an ARMA(p,q) equation was selected and parameters were estimated using standard time-series analysis on the range of available data.[7] The fit equation is of the form:

$$Y_t = \mu + \varphi_1(Y_{t-1} - \mu) + \dots + \varphi_p\left(Y_{t-p} - \mu\right) + \varepsilon_t + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q}.$$

In this equation, Y_t represents the variable being modeled in year *t,* μ represents the estimated mean from the equation, and ε_t represents the random error in year *t*. In order to center the variable around the intermediate alternative, the modified equation is:

$$Y_t = Y_t^{TR} + \varphi_1 y_{t-1} + \dots + \varphi_p y_{t-p} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q}.$$

Y_t^{TR} represents the variable's projected value from the intermediate assumptions of the 2023 Trustees Report in year *t*, and y_t represents the deviation of the variable's value from the intermediate assumptions of the 2023 Trustees Report value in year *t* (i.e., $y_t = Y_t - Y_t^{TR}$).

The equation for performing simulations with parameter uncertainty is:

$$Y_t = Y_t^{PU} + \varphi_1(Y_{t-1} - Y_{t-1}^{PU}) + \dots + \varphi_p(Y_{t-p} - Y_{t-p}^{PU}) + \varepsilon_t + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q}.$$

In this equation, Y_t^{PU} is the mean, which varies from one simulation to another, and around which each simulation is centered. This equation can be rewritten as:

$$Y_t = Y_t^{PU} + \varphi_1 y_{t-1}^{PU} + \dots + \varphi_p y_{t-p}^{PU} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q},$$

$$\text{where } y_t^{PU} = Y_t - Y_t^{PU}.$$

[7] Variables that deviate from this pattern are three economic variables (the unemployment rate, inflation rate, and the real interest rate), which are jointly estimated in a vector autoregressive (VAR) model (section III.D), and the real average covered wage, which is modeled as the dependent variable regressed on lagged values of the unemployment rate (section III.E).

A. FERTILITY

The *total fertility rate* is the sum of age-specific birth rates[8] for girls and women aged 14 through 49. Thus, the total fertility rate may be interpreted as the average number of children that would be born to a woman in her lifetime if she were to experience, at each age of her life, the birth rate observed in, or assumed for, a specified year, and if she were to survive the entire childbearing period.

Historical values for the total fertility rate in the U.S. for 1917 through 1979 are available from the National Center for Health Statistics (NCHS).[9] Values for total fertility rate in the U.S. for 1980 through 2021 are calculated from NCHS birth data and the U.S. Census Bureau[10] population estimates. Over the historical period prior to 1990, the total fertility rate ranged from a minimum of 1.74 in 1976 to a maximum of 3.68 in 1957. From 1990 through 2007, it remained relatively stable, fluctuating between 1.97 and 2.12, but has dropped significantly since the start of the Great Recession. The rate was 1.66 in 2021.

For the total fertility rate, an ARMA(4,1) equation was selected and parameters were estimated using the entire range of data. Figure III.1 presents the actual and fitted values. The R-squared value is 0.98. The equation is:

$$F_t = F_t^{PU} + 1.96 f_{t-1}^{PU} - 1.46 f_{t-2}^{PU} + 0.88 f_{t-3}^{PU} - 0.41 f_{t-4}^{PU} + \varepsilon_t - 0.60\varepsilon_{t-1},$$

where F_t is the total fertility rate in year *t*; F_t^{PU} is the mean (which varies from one simulation to another, and around which each simulation is centered); $f_{t-n}^{PU} = F_{t-n} - F_{t-n}^{PU}$; and ε_t represents the random error in year *t*.

[8] Age-specific birth rates are defined as the ratio of: (1) the number of live births to mothers of a specified age, to (2) the midyear female population of that age.

[9] See www.cdc.gov/nchs.

[10] See www.census.gov.

Figure III.1 – Total Fertility Rate, Calendar Years 1917-2021

B. MORTALITY

The annual rate of decrease in the *central death rate*[11] (which is sometimes referred to as the annual rate of improvement in mortality) is calculated as the negative of the percent change in the central death rate for a given year. Thus, a positive value represents a decrease in the central death rate from one year to the next.

Central death rates were calculated for 42 age-sex groups (under 1, 1-4, 5-9, 10-14, …, 85-89, 90-94, and 95+; male and female) for the period 1900 through 2020.[12] Data for the annual numbers of deaths and the U.S. resident population for the population under age 65 are from the National Center for Health Statistics and the U.S. Census Bureau, respectively. For the population aged 65 and older, annual deaths and enrollments are from the Centers for Medicare & Medicaid Services.

An AR(1) equation was selected for the annual rate of decrease in the central death rate for each age-sex group. The general form of the equation is:

$$M_{k,t} = M_{k,t}^{PU} + \phi_k\left(m_{k,t-1}^{PU}\right) + \varepsilon_{k,t},$$

where $M_{k,t}$ is the rate of mortality improvement for age group k in year t; $M_{k,t}^{PU}$ is the mean (which varies from one simulation to another, and around which each simulation is centered); $m_{k,t-n}^{PU} = M_{k,t-n} - M_{k,t-n}^{PU}$; and $\varepsilon_{k,t}$ represents the random error for age group k in year t. Appendix F contains the estimates of the parameters ϕ_k for each age group k.

A Cholesky decomposition was performed using the residuals from the 42 fitted equations. Appendix C discusses this technique. The Cholesky matrix used was $42\ x\ 42$ with the age groups in ascending order with alternating male and female groups.

[11] The central death rate is defined as the annual number of deaths for a particular group divided by the midyear population of that group.

[12] A detailed description of the methodology used in calculating death rates by age and sex can be found in Actuarial Study No. 120. See www.ssa.gov/oact/NOTES/pdf_studies/study120.pdf.

C. IMMIGRATION

The OSM models five categories of immigration flows:

1. Lawful permanent resident (LPR) new arrival immigration: Flow of persons who enter the Social Security area and are granted LPR status.
2. Adjustments of Status: Flow of persons who are already in the Social Security area and adjust their status to become LPRs.
3. Legal emigration: Flow of LPRs and citizens who leave the Social Security area population.
4. Other-than-LPR immigration: Flow of persons who enter the Social Security area and stay to the end of the year without being granted LPR status, such as undocumented immigrants, and foreign workers and students entering with temporary visas.
5. Other-than-LPR emigration: Flow of other-than-LPR immigrants who leave the Social Security area population or who adjust their status to become LPRs.

The first four of these categories are discussed in the following four subsections. The last category, other-than-LPR emigration, does not have its own specific equation because the flow of those leaving the Social Security area is modeled using rates of departure, which do not vary stochastically.

1. LPR New Arrival Immigration

LPR immigration is defined as the flow of persons lawfully admitted for permanent residence into the United States, whether as a new arrival or as an adjustment of status.[13] The level of LPR immigration largely depends on legislation that serves to define and establish limits for certain categories of immigrants. The Immigration Act of 1990, which is currently the legislation in force, establishes limits for three classes of immigrants: family-sponsored preferences, employment-based preferences, and diversity immigrants. However, no numerical limits currently exist for immediate relatives of U.S. citizens.

Historical data for LPR immigration for years 1901 through 2020 are from the U.S. Citizenship and Immigration Services. LPR immigration averaged nearly one million per year from 1900 through 1914, then decreased substantially to about 23,000 in 1933. Since the mid-1940s, LPR immigration has generally increased through about the year 2000. From 2000 – 2019, it remained generally steady, averaging a little over one million per year. Due to the COVID-19 pandemic, LPR immigration dropped to just a little over 700,000 persons in 2020.

LPR immigration, as defined above, includes both new arrivals and adjustments of status. The drop in total LPR immigration in 2020 is mostly due to the drop in new arrivals. These two categories are modeled with separate equations. For new arrival LPR immigration, an ARMA(1,1) equation was selected and parameters were estimated using historical data from 1991 – 2020.

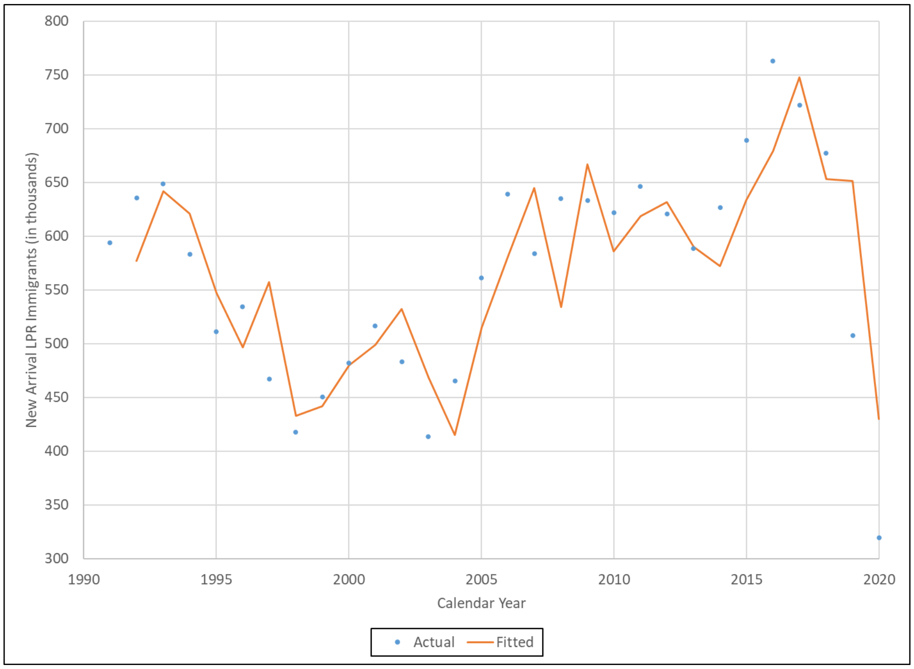
[13] For more detailed information, refer to the Yearbook of Immigration Statistics at www.dhs.gov/immigration-statistics/yearbook.

Figure III.2 presents the actual and fitted values. The R-squared value is 0.69. The equation is:

$$L_t = L_t^{PU} + 0.72 l_{t-1}^{PU} + \varepsilon_t + 0.59\varepsilon_{t-1},$$

where L_t is the new arrival LPR immigration in year *t*; L_t^{PU} is the mean (which varies from one simulation to another, and around which each simulation is centered); $l_{t-n}^{PU} = L_{t-n} - L_{t-n}^{PU}$; and ε_t represents the random error in year *t*.

Figure III.2 – LPR New Arrival Immigration, Calendar Years 1991-2020



2. Adjustments of Status

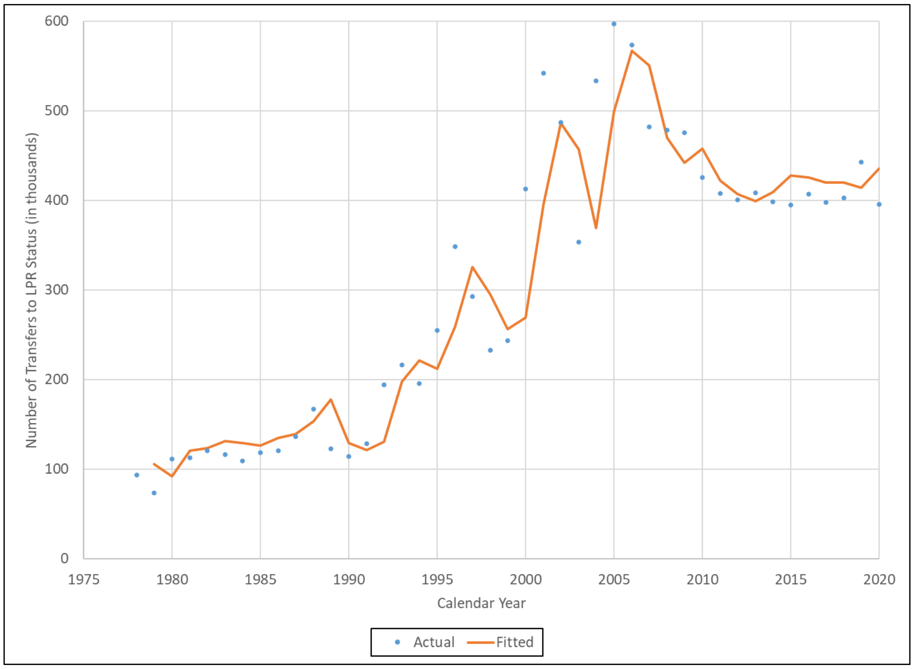
Adjustments of status is defined as the flow of persons within the United States who adjust from other-than-LPR status to LPR status.

As mentioned above, adjustments of status are a subset of LPR immigration, with the other subset being LPR new arrival immigration. The transfer rate is the percentage of the beginning-of-year other-than-LPR population that adjusts status to become LPRs. Having the regression based on the transfer rate, rather than adjustments of status values themselves, ensures that large numbers of adjustments of status will not cause the other-than-LPR population to become negative. To model the transfer rate, an AR(1) model was selected and parameters were estimated using historical data from 1978 – 2020. Figure III.3 presents the actual and fitted adjustments of status levels. The R-squared value is 0.49. The equation is:

$$S_t = S_t^{PU} + 0.70 s_{t-1}^{PU} + \varepsilon_t,$$

where S_t is the LPR immigration transfer rate in year *t*; S_t^{PU} is the mean (which varies from one simulation to another, and around which each simulation is centered); $s_{t-n}^{PU} = S_{t-n} - S_{t-n}^{PU}$; and ε_t represents the random error in year *t*.

Figure III.3 – Adjustments to LPR Status, Calendar Years 1978-2020



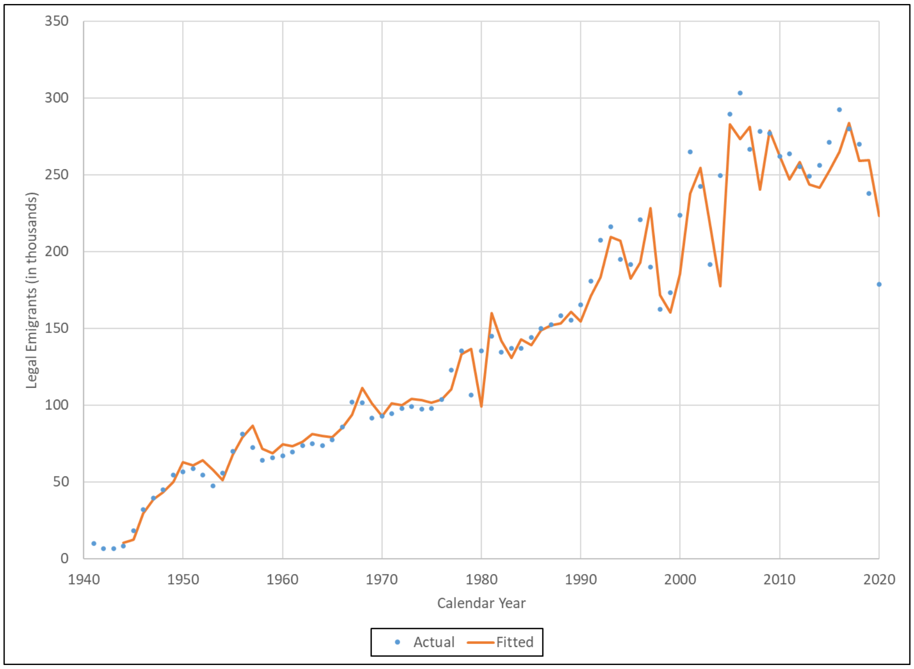
3. Legal Emigration

Legal emigration is defined as the flow of citizens and LPRs who leave the United States and are no longer considered to be a part of the Social Security area population. Although annual emigration data are not collected in the United States, we estimate, using U.S. Census Bureau estimates as a guide, that the level of legal emigration for the past century roughly totaled one-fourth of the level of LPR immigration. The legal emigration rate is the percentage of the beginning-of-year LPR plus citizen population who emigrate during the year. Having the regression based on the emigration rate, rather than legal emigrant values themselves, ensures that large numbers of emigrants will not cause the total population to become negative. Using annual legal emigration values set at one-fourth of the LPR immigration values for 1941 – 2020, an AR(3) equation was selected based on these values divided by the LPR plus citizen population at the beginning of each year and parameters were estimated. Figure III.4 presents the actual and fitted emigration levels. The R-squared value is 0.95. The equation is:

$$E_t = E_t^{PU} + 1.27e_{t-1}^{PU} - 0.63e_{t-2}^{PU} + 0.23e_{t-3}^{PU} + \varepsilon_t,$$

where E_t is the legal emigration in year *t*; E_t^{PU} is the mean (which varies from one simulation to another, and around which each simulation is centered); $e_{t-n}^{PU} = E_{t-n} - E_{t-n}^{PU}$; and ε_t represents the random error in year *t*.

Figure III.4 – Legal Emigration, Calendar Years 1941-2020



4. Other-than-LPR Immigration

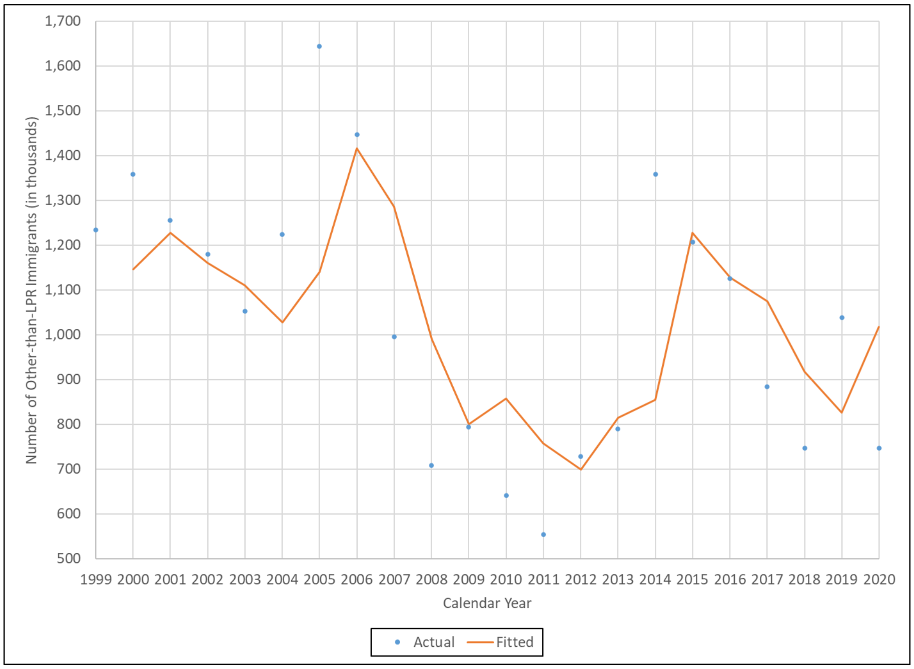
Other-than-LPR immigration is defined as the flow of persons into the United States who do not meet the above definition for LPR immigration. This includes unauthorized persons and those entering legally but without having LPR status, such as legal temporary workers and students.

Other-than-LPR immigration data is derived by subtracting LPR new arrivals from foreign-born new arrivals, based on the American Community Survey. An AR(1) model was selected and parameters were estimated using historical data from 1999 – 2020. Figure III.5 presents the actual and fitted values. The R-squared value is 0.42. The equation is:

$$O_t = O_t^{PU} + 0.66o_{t-1}^{PU} + \varepsilon_t,$$

where O_t is the other-than-LPR immigration in year *t*; O_t^{PU} is the mean (which varies from one simulation to another, and around which each simulation is centered); $o_{t-n}^{PU} = O_{t-n} - O_{t-n}^{PU}$; and ε_t represents the random error in year *t*.

Figure III.5 – Other-than-LPR Immigration, Calendar Years 1999-2020



D. UNEMPLOYMENT RATE, INFLATION RATE, AND REAL INTEREST RATE

1. Description of the Vector Autoregression Process

The unemployment rate, inflation rate (growth rate in the CPI-W), and real interest rate are simulated together using a vector autoregression (VAR), in order to capture the relationship among these three variables that economic theory suggests are related.[14] In a vector autoregression, each variable is regressed on prior-period values of all variables. The general form of a VAR process is

$$\mathbf{Y}_t = \boldsymbol{\mu} + \boldsymbol{\Phi}_1(\mathbf{Y}_{t-1} - \boldsymbol{\mu}) + \cdots + \boldsymbol{\Phi}_p\left(\mathbf{Y}_{t-p} - \boldsymbol{\mu}\right) + \boldsymbol{\varepsilon}_t.$$

In this equation, $\mathbf{Y}_t$ is a vector of dimension *n* (the number of variables in the model), such that its i^{th} element is the value of the i^{th} variable in year *t,* i.e., Y_t^i; $\boldsymbol{\mu}$ is the vector of means, such that its i^{th} element is the mean of the i^{th} variable, i.e., $\mu^i = \text{mean}\left(Y^i\right)$; p is the longest lag (highest number of prior periods) used; $\boldsymbol{\Phi}_k, k = 1, \dots, p$ is the matrix of autoregressive coefficients for lag *k*, such that its element φ_k^{ij} is the coefficient of *k*-years-lagged j^{th} variable in the equation for the i^{th} variable; and $\boldsymbol{\varepsilon}_t$ is the random error in year *t*. The vector equation is equivalent to a system of equations such that the equation for the i^{th} variable is

$$Y_t^i = \mu^i + \sum_{k=1}^{p} \sum_{j=1}^{n} \varphi_k^{ij}\left(Y_{t-k}^j - \mu^j\right) + \varepsilon_t^i,$$

Where *n* is the number of variables, *p* is the number of lags (prior periods) used, Y_t^i is the value of the i^{th} variable in year *t,* μ^i is the mean of the i^{th} variable, φ_k^{ij} is the coefficient of *k*-years-lagged j^{th} variable in the equation for the i^{h} variable, and ε_t^i is the random error in year *t* associated with the i^{th} variable.

For simulating projected values, just like in the single-variable equations in the preceding sections, the means are replaced by values from the intermediate alternative of the deterministic model in the case without parameter uncertainty, and with the simulated central tendencies, which vary from one simulation to another, in the case with parameter uncertainty.

In our case, there are three variables—the unemployment rate, the inflation rate, and the real interest rate. Vector autoregressions of different prior-period lengths were tested and it was determined that a vector autoregression including two prior years provided a good fit. Thus, using the notation above, *n* = 3 and *p* = 2. The historical period used for estimating the model is 1961 to 2020.

[14] Foster (1994) suggested that a multivariate approach might capture a more appropriate range of variability for these economic variables.

2. Variables

a) Unemployment Rate

The *unemployment rate* is the number of unemployed persons seeking work as a percentage of the civilian labor force. Historical values are published by the Bureau of Labor Statistics (BLS).[15] The annual levels are averages of the 12 corresponding monthly rates.

Between 1961 and 2020, the unemployment rate averaged 6.3 percent, reaching a low of 3.5 percent in 1969 and a high of 9.7 percent in 1982. In the VAR equation, the unemployment rates are transformed to log-odds ratios[16] to bound the values of the unemployment rate between 0 and 100 percent.

In the equations that follow, the transformed unemployment rate in year *t* is denoted U_t, its central tendency for a given simulation in year *t* is denoted U_t^{PU}, and its deviation from the central tendency is denoted u_t^{PU} (i.e., $u_t^{PU} = U_t - U_t^{PU}$). In the case without parameter uncertainty, the value from the intermediate alternative of the deterministic model, U_t^{TR}, is used for the central tendency in all simulations, and $u_t = U_t - U_t^{TR}$.

b) Inflation Rate

The *inflation rate* is defined as the annual growth rate in the Consumer Price Index for Urban Wage Earners and Clerical Workers (CPI-W). BLS publishes historical values for the CPI-W.[17] BLS periodically introduces improvements to the CPI-W that affect its annual growth rate but does not revise earlier values. Consequently, OCACT has adjusted the CPI-W by back-casting the effects of the improvements on earlier values so that rates are comparable over the historical and projection periods.

Over the historical period from 1961 to 2020, the adjusted inflation rate ranged from a low of -0.7 percent in 2009 to a high of 11.1 percent in 1980. The estimation period includes a period of high and rising inflation rates in the 1970s, followed by a tightened monetary policy and a decrease in the inflation rate in the early 1980s, and eventually a period of explicit inflation targeting in the 21st century. In the VAR equation, a logarithmic transformation[18] is applied to the adjusted inflation rates so that the underlying rates have a lower bound. Instead of a traditional log-transformation of the inflation rate series, 3.0 percent is added to the inflation rate series prior to the log-transformation. This gives the inflation rates a lower bound of -3.0 percent.

In the equations that follow, the transformed inflation rate in year *t* is denoted I_t, its central tendency for a given simulation in year *t* is denoted I_t^{PU}, and its deviation from the central tendency

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[15] See www.bls.gov/cps/home.htm.

[16] $U_t = \log[RU_t / (1 - RU_t)]$, where RU_t is the unemployment rate in year *t* expressed as a decimal and U_t is the transformed unemployment rate.

[17] See www.bls.gov/cpi.

[18] $I_t = \log(\pi_t + 0.03)$ where π_t is the percent change in the adjusted inflation rate in year *t* expressed as decimals.

is denoted i_t^{PU} (i.e., $i_t^{PU} = I_t - I_t^{PU}$). In the case without parameter uncertainty, the value from the intermediate alternative of the deterministic model, I_t^{TR}, is used for the central tendency in all simulations, and $i_t = I_t - I_t^{TR}$.

c) Real Interest Rate

All securities held by the OASI and DI Trust Funds are special issues (i.e., securities issued only to the trust funds) that are issued by the Federal Government. Historical data on actual nominal interest rates of new purchases of these securities are published by OCACT.[19] The nominal interest rate on new purchases of these securities for a given month is set equal to the average market yield on all marketable federal obligations that are not callable and do not mature within the next 4 years.[20] Annual nominal interest rates are the average of the 12 monthly rates, which, in practice, are compounded semiannually.[21] The *real interest rate* earned on these obligations in year *t* is equal to the annual (compounded) *nominal yield* in year *t*-1 divided by the inflation rate in year *t*.

Looking at the period from 1961 to 2020, real interest rates on new purchases of special issues were at relatively low levels in the 1960s and 1970s. Rates rose to much higher levels in the 1980s, as investors demanded higher risk premiums for increased uncertainty surrounding the unexpectedly high rates of inflation. Since then, the rate of inflation and the real interest rate have declined to lower levels.

In the equations that follow, the real interest rate in year *t* is denoted R_t, its central tendency for a given simulation in year *t* is denoted R_t^{PU}, and its deviation from the central tendency is denoted r_t^{PU} (i.e., $r_t^{PU} = R_t - R_t^{PU}$). In the case without parameter uncertainty, the value from the intermediate alternative of the deterministic model, R_t^{TR}, is used for the central tendency in all simulations, and $r_t = R_t - R_t^{TR}$.

3. Equations and Fits

The equation for the transformed unemployment rate with parameter uncertainty is:

$$U_t = U_t^{PU} + 1.10u_{t-1}^{PU} - 0.44u_{t-2}^{PU} + 0.09i_{t-1}^{PU} + 0.08i_{t-2}^{PU} - 0.91r_{t-1}^{PU} + 0.11r_{t-2}^{PU} + \varepsilon_t^U.$$

In this equation, U_t^{PU} is the central tendency of the transformed unemployment rate for a given simulation; u_t^{PU} is the deviation of the transformed unemployment rate from its central tendency in year *t* (i.e., $u_t^{PU} = U_t - U_t^{PU}$); i_t^{PU} is the deviation of the transformed inflation rate from its central tendency in year *t* (i.e., $i_t^{PU} = I_t - I_t^{PU}$); r_t^{PU} is the deviation of the real interest rate from

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[19] See www.ssa.gov/OACT/ProgData/newIssueRates.html.

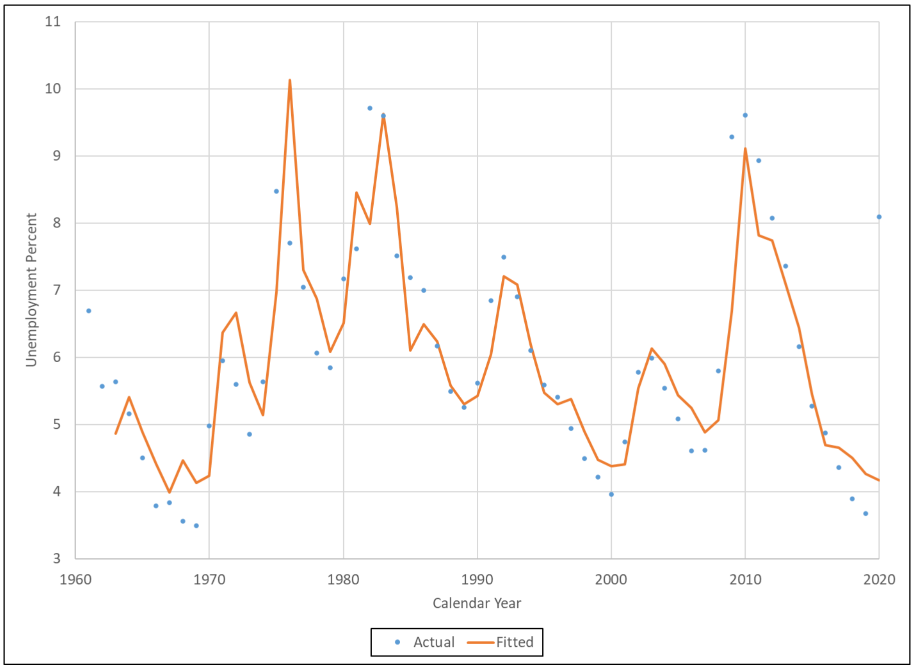
[20] For more details on the history of trust fund investment policy, see Actuarial Note 142, *Social Security Trust Fund Investment Policies and Practices*, by Jeff Kunkel, at www.ssa.gov/OACT/NOTES/pdf_notes/note142.pdf.

[21] For example, the annualized nominal yield on a special issue with a 5.0-percent nominal interest rate is equal to $(1 + 0.05 / 2)^2 - 1 = 5.06$ percent.

its central tendency in year *t* (i.e., $r_t^{PU} = R_t - R_t^{PU}$); ε_t^U is the random error associated with *U* in year *t*.

For the unemployment rate equation, the R-squared value is 0.71. The actual and fitted values are shown in figure III.6.

Figure III.6 – Unemployment Rate, Calendar Years 1961-2020



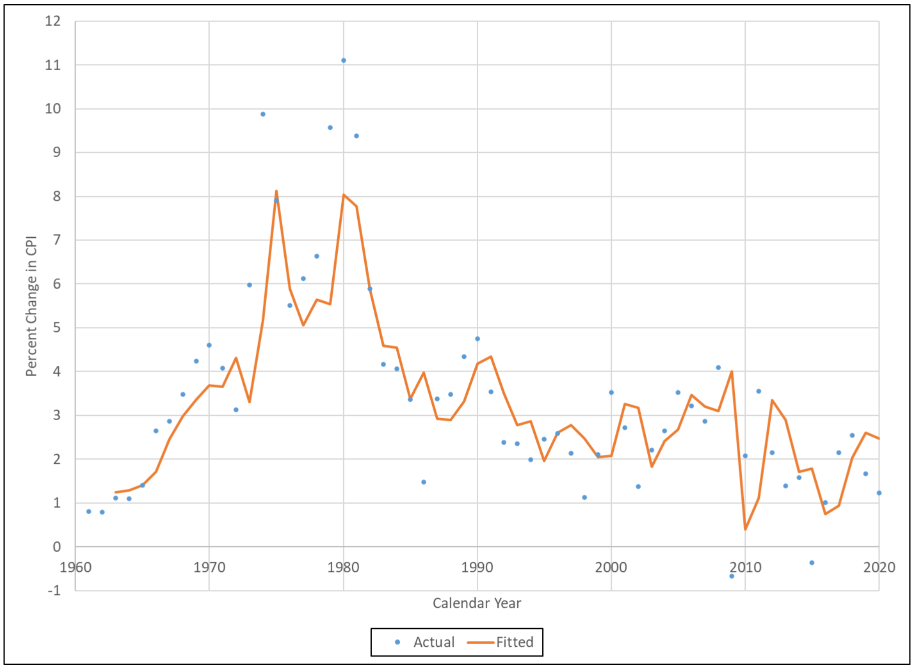
The equation for the transformed inflation rate with parameter uncertainty is:

$$I_t = I_t^{PU} - 0.21u_{t-1}^{PU} + 0.16u_{t-2}^{PU} + 0.30i_{t-1}^{PU} + 0.53i_{t-2}^{PU} - 6.82r_{t-1}^{PU} + 6.57r_{t-2}^{PU} + \varepsilon_t^I.$$

In this equation, I_t^{PU} is the central tendency of the inflation rate for a given simulation; u_t^{PU} is the deviation of the transformed unemployment rate from its central tendency in year *t* (i.e., $u_t^{PU} = U_t - U_t^{PU}$); i_t^{PU} is the deviation of the transformed inflation rate from its central tendency in year *t* (i.e., $i_t^{PU} = I_t - I_t^{PU}$); r_t^{PU} is the deviation of the real interest rate from its central tendency in year *t* (i.e., $r_t^{PU} = R_t - R_t^{PU}$); ε_t^I is the random error associated with *I* in year *t*.

For the inflation rate equation, the R-squared value was 0.56. The actual and fitted values are shown in figure III.7.

Figure III.7 – Inflation Rate (CPI-W), Calendar Years 1961-2020



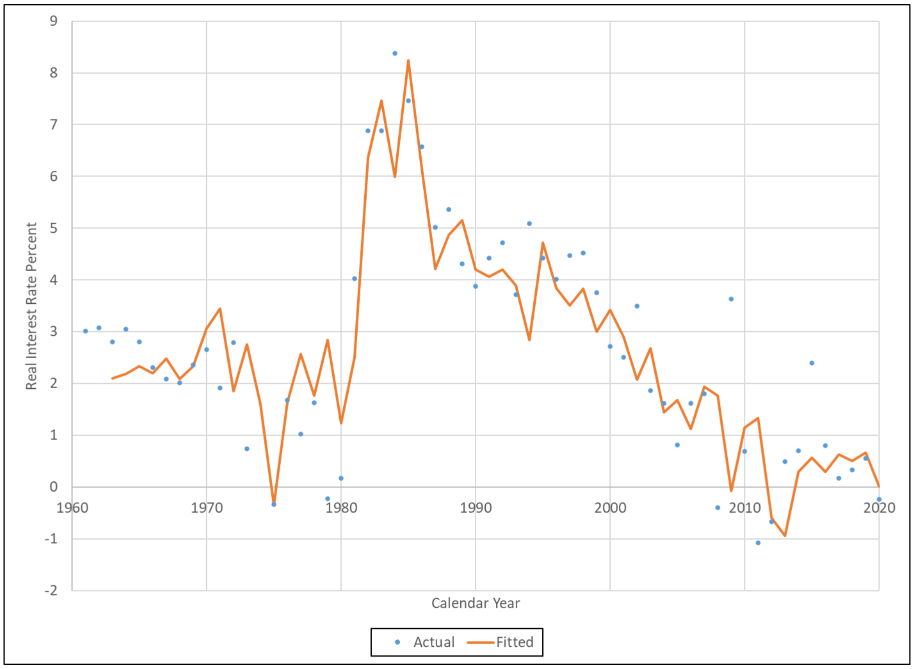
The equation for the real interest rate with parameter uncertainty is:

$$R_t = R_t^{PU} + 0.01u_{t-1}^{PU} - 0.01u_{t-2}^{PU} + 0.04i_{t-1}^{PU} - 0.02i_{t-2}^{PU} + 1.28r_{t-1}^{PU} - 0.43r_{t-2}^{PU} + \varepsilon_t^R.$$

In this equation, R_t^{PU} is the central tendency of the real interest rate for a given simulation; u_t^{PU} is the deviation of the transformed unemployment rate from its central tendency in year *t* (i.e., $u_t^{PU} = U_t - U_t^{PU}$); i_t^{PU} is the deviation of the transformed inflation rate from its central tendency in year *t* (i.e., $i_t^{PU} = I_t - I_t^{PU}$); r_t^{PU} is the deviation of the real interest rate from its central tendency in year *t* (i.e., $r_t^{PU} = R_t - R_t^{PU}$); ε_t^R is the random error associated with *R* in year *t*.

For the real interest rate equation, the R-squared value is 0.71. The actual and fitted values are shown in figure III.8.

Figure III.8 – Real Interest Rate, Calendar Years 1961-2020



E. REAL AVERAGE COVERED WAGE

The *real average covered wage* is defined as the ratio of the average nominal OASDI covered wage to the adjusted CPI. Because of the expansion of covered employment, the historical annual growth rate in the real average covered wage differs from the annual growth rate in the real average economy-wide wage. In the future, however, the annual growth rates in the two measures are expected to be approximately equal, since projected coverage changes are insignificant. Hence, the historical variation of the annual percent change in the real average economy-wide wage is used to model the future variation of the annual percent change in the real average covered wage.

The real average economy-wide wage is the ratio of the average nominal wage to the adjusted CPI. The nominal wage is the ratio of wages and salaries as published by the Bureau of Economic Analysis' (BEA) National Income and Product Accounts (NIPA) to total wage employment. Total wage employment is the sum of civilian wage employment, as published by the BLS from its Household Survey, and total U.S. Armed Forces from the Census Bureau. The BLS periodically introduces improvements to its employment data but does not revise earlier data. However, the BLS has developed adjustment factors to improve the comparability of employment data with earlier years.[22] OCACT uses these factors to adjust the wage employment data.

The formula for calculating the annual percent change in the real average wage, given a nominal wage series, is:

$$W_t = \frac{\left(\frac{NW_t}{NW_{t-1}}\right)}{\left(\frac{CPI_t}{CPI_{t-1}}\right)} - 1.$$

W_t is the annual percent change in the real average wage expressed in decimals in year t; NW_t is the level of the nominal average wage in year t; and CPI_t is the level of the CPI in year t.

The annual percent changes in the real economy-wide wage are modeled as a function of the current unemployment rate and the unemployment rate for the previous year, expressed as log-odds ratios. The coefficients were estimated using data from 1968 to 2020. The R-squared value is 0.10. The actual and fitted values are shown in figure III.9. The equation is:

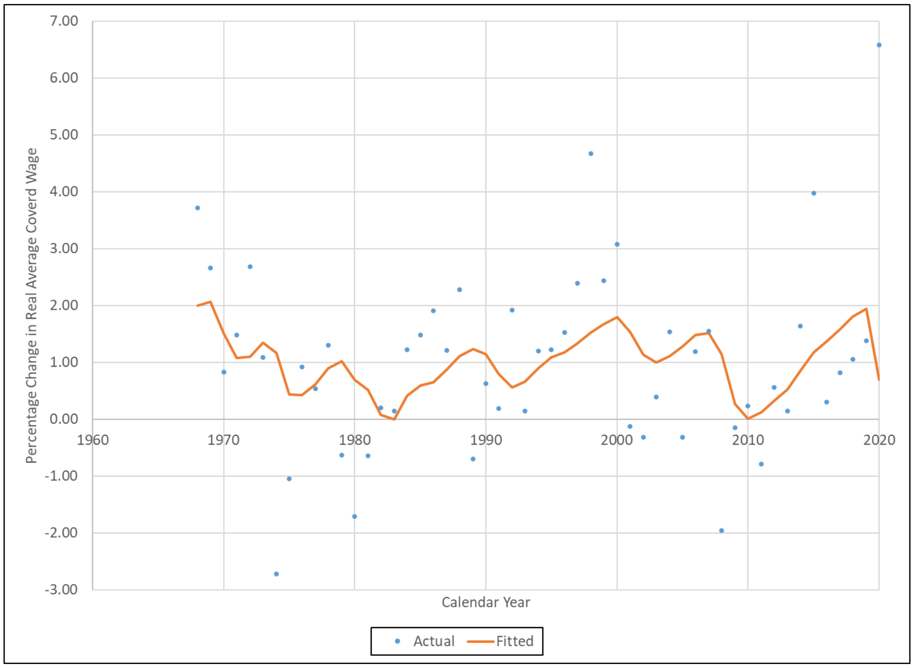
$$W_t = W_t^{PU} - 1.53u_t^{PU} - 0.39u_{t-1}^{PU} + \varepsilon_t.$$

In this equation, W_t represents the percent change in the real average covered wage in year t; W_t^{PU} is the mean around which each simulation is centered, which varies from one simulation to another; u_t^{PU} represents the deviation of the log-odds transformed unemployment rate from the mean log-odds transformed unemployment rate under parameter uncertainty (which also varies from one

[22] For a detailed description of the methodology, refer to the article titled Creating Comparability in CPS Employment Series at www.bls.gov/cps/cpscomp.pdf.

simulation to another) in year *t* (i.e., $u_t^{PU} = U_t - U_t^{PU}$); and ε_t represents the random error in year t.

Figure III.9 – Real Average Covered Wage (Percent Change), Calendar Years 1968-2020



F. DISABILITY INCIDENCE RATE

The *disability incidence rate* for a given year is the proportion of the exposed population at the beginning of that year that becomes newly entitled to disability benefits during the year. The exposed population consists of workers who are disability insured but not entitled to disability benefits. The historical disability incidence rates used to fit the equations are age-adjusted to the exposed population in year 2000. The age-adjusted disability incidence rates (male and female) are the rates that would occur using the disability exposed population as of January 1, 2000, if that population were to experience the observed or assumed age-sex specific disability incidence rates in the selected year.

Data on disability incidence were obtained from SSA administrative records, and the age-adjusted disability incidence rates were computed by OCACT. Over the historical period from 1970 to 2022, disability incidence rates varied widely due to changes in legislation and program administration as well as economic and demographic factors.

The equations for disability incidence rates were selected separately for men and women. The disability incidence rates are expressed as log-odds ratios to bound the values between 0 and 100 percent. For the remainder of this section, all references to the equations for the disability incidence rates refer to the transformed rates.

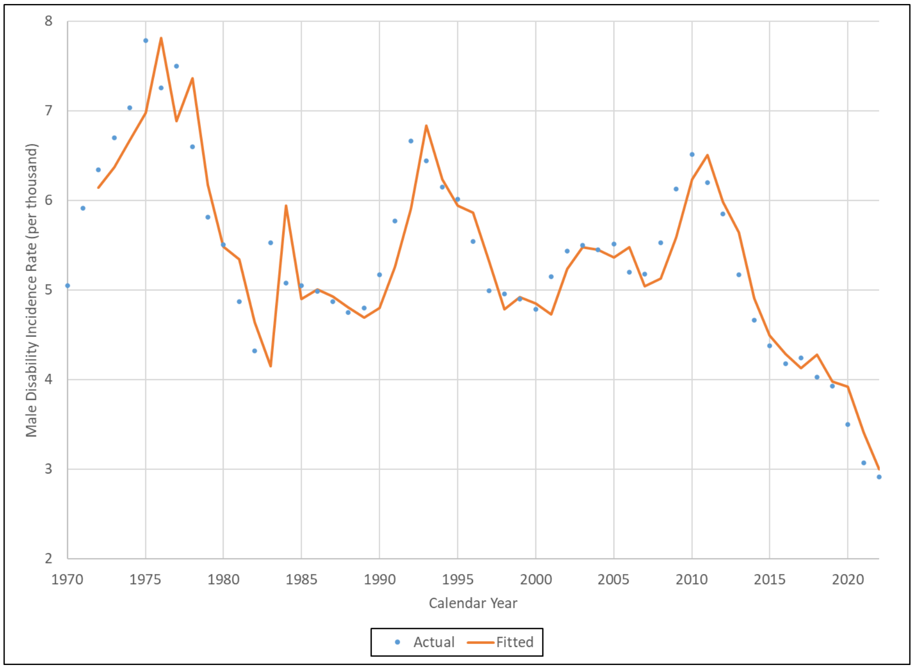
The male and female disability incidence rates were modeled individually as AR(2) processes. The actual and fitted values for men and women are shown in figures III.10 and III.11, respectively. The R-squared values for the male and female disability incidence rate equations are both 0.86.

The age-adjusted male disability incidence rate equation is:

$$DIM_t = DIM_t^{PU} + 1.29dim_{t-1}^{PU} - 0.34dim_{t-2}^{PU} + \varepsilon_t,$$

where DIM_t is the age-adjusted male disability incidence rate in year *t*; DIM_t^{PU} is the mean (which varies from one simulation to another, and around which each simulation is centered); $dim_{t-n}^{PU} = DIM_{t-n} - DIM_{t-n}^{PU}$; and ε_t represents the random error in year *t*.

Figure III.10 – Male Disability Incidence Rate, Calendar Years 1970-2022

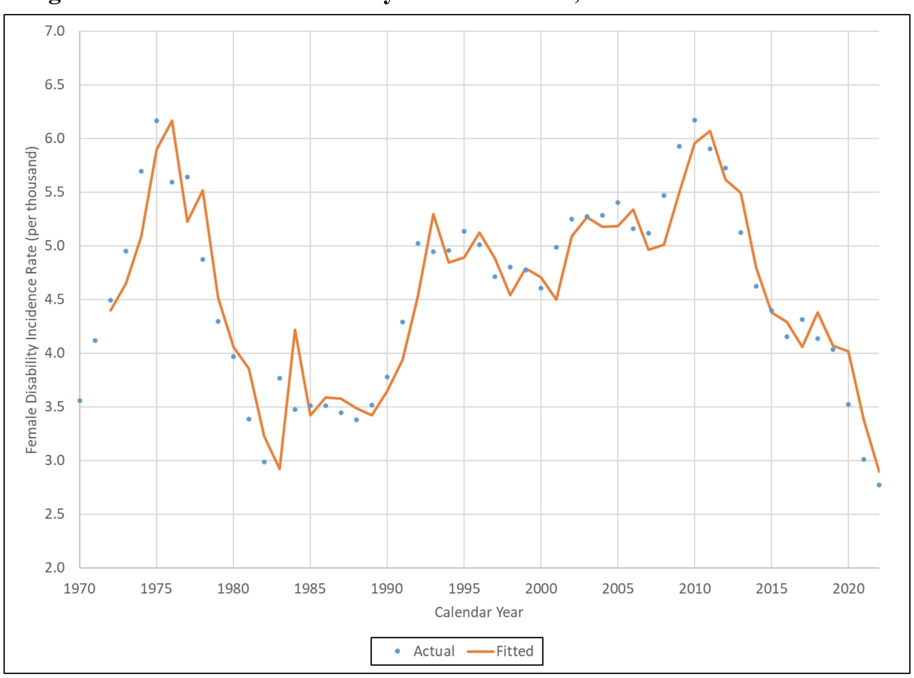


The age-adjusted female disability incidence rate equation is:

$$DIF_t = DIF_t^{PU} + 1.35dif_{t-1}^{PU} - 0.44dif_{t-2}^{PU} + \varepsilon_t,$$

where DIF_t is the age-adjusted female disability incidence rate in year *t*; DIF_t^{PU} is the mean (which varies from one simulation to another, and around which each simulation is centered); $dif_{t-n}^{PU} = DIF_{t-n} - DIF_{t-n}^{PU}$; and ε_t represents the random error in year *t*.

Figure III.11 – Female Disability Incidence Rate, Calendar Years 1970-2022



G. DISABILITY RECOVERY RATE

The *disability recovery rate* for a given year is the proportion of disabled-worker beneficiaries whose disability benefits terminate as a result of their recovery from disability. The age-adjusted disability recovery rates (male and female) are the rates that would occur in the in-current-payment population as of January 1, 2000, if the population were to experience the observed or assumed age-sex specific disability recovery rates in the selected year.

Data on disability recovery were obtained from SSA administrative records, and OCACT computed the age-adjusted disability recovery rates. Over the historical period from 1970 to 2022, there has been substantial variation in the age-adjusted disability recovery rates. The rate of recovery is affected by budget appropriations for continuing disability reviews. Changes in law have also caused variation. For example, the age-adjusted disability recovery rate for men increased from 10.3 per thousand in 1996 to 24.5 per thousand in 1997, largely as a result of the effects of a provision in Public Law 104-121 that prohibited benefit payments to individuals for whom drug addiction and/or alcoholism was material to the determination of disability.

The equations for disability recovery rates are modeled separately for men and women. The historical disability recovery rates used to fit the equations are age-adjusted to the 2000 in-current-payment population. The disability recovery rates are expressed as log-odds ratios to bound the values between 0 and 100 percent. For the remainder of this section, all equations for the disability recovery rates refer to the transformed rates. Due to the frequent changes in the law, the period considered has been narrowed to 1984 through 2022. The value for 1997 is excluded in the development of the equation due to the change in the law described above.

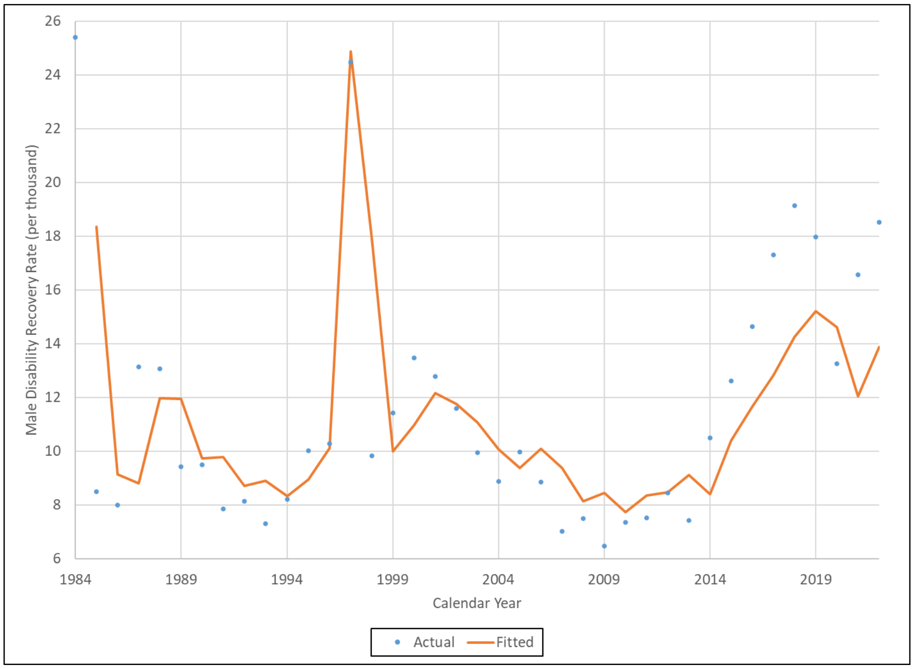
The AR(1) model provides the best fit of the models that have been tested. The actual and fitted values for males and females are shown in figures III.12 and III.13, respectively. The R-squared values for the male and female disability recovery rate equations are 0.55 and 0.57, respectively.

The male age-adjusted disability recovery rate equation is:

$$DRM_t = DRM_t^{PU} + 0.62 drm_{t-1}^{PU} + \varepsilon_t,$$

where DRM_t is the age-adjusted male disability recovery rate in year *t*; DRM_t^{PU} is the mean (which varies from one simulation to another, and around which each simulation is centered); $drm_{t-n}^{PU} = DRM_{t-n} - DRM_{t-n}^{PU}$; and ε_t represents the random error in year *t*.

Figure III.12 – Male Disability Recovery Rate, Calendar Years 1984 – 2022

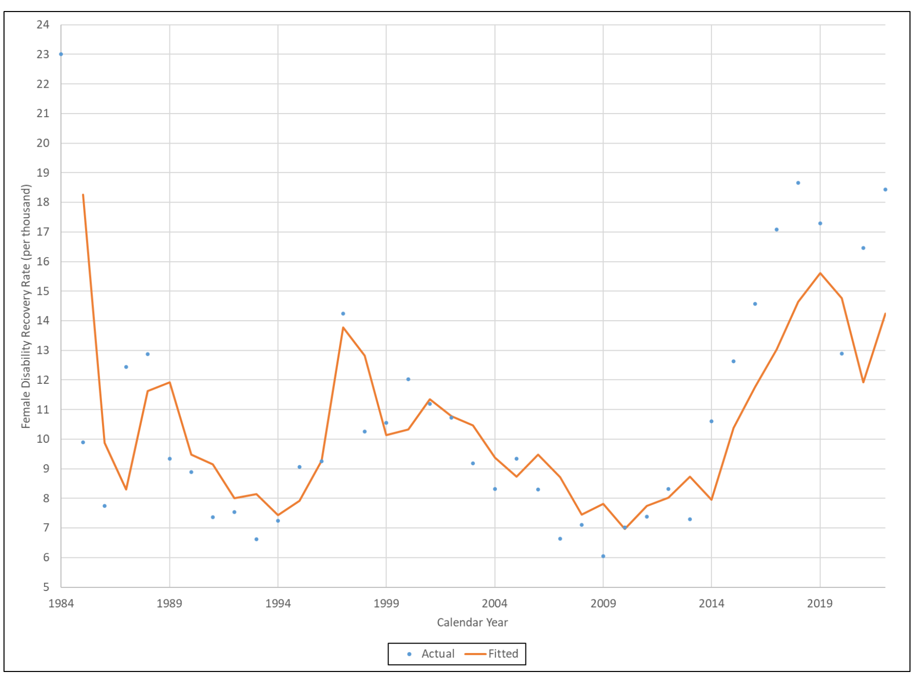


The female age-adjusted disability recovery rate equation is:

$$DRF_t = DRF_t^{PU} + 0.71drf_{t-1}^{PU} + \varepsilon_t,$$

where DRF_t is the age-adjusted female disability recovery rate in year *t*; DRF_t^{PU} is the mean (which varies from one simulation to another, and around which each simulation is centered); $drf_{t-n}^{PU} = DRF_{t-n} - DRF_{t-n}^{PU}$; and ε_t represents the random error in year *t*.

Figure III.13 – Female Disability Recovery Rate, Calendar Years 1984 – 2022



IV. RESULTS

This chapter presents the OSM simulation results in three sections. The results presented in this chapter are based on two sets of 5,000 simulations; one set is with parameter uncertainty and the other set is without parameter uncertainty. The first section describes the OSM-simulated probability distributions for the equations presented in chapter III. In order to better illustrate the different approaches to assessing uncertainty, frequency interval bounds from the OSM results are compared with the values assumed in the low-cost and high-cost deterministic alternatives from the 2023 Trustees Report. The second section contains selected actuarial estimates including annual trust fund ratios and balances, as well as summary actuarial measures (e.g., actuarial balances and cost rates). The third section illustrates the sensitivity of the OSM to variations in the assumptions.

A. EQUATION RESULTS

The tables and figures in the following subsections display the OSM results for the equations presented in chapter III. The three rows of values shown in the tables are the level in the 75th projection year (2097), the average level over the entire 75-year projection period (2023-97), and the average over the final 50 years of the projection period (2048-97).[23] The corresponding columns for the three rows are the value from the 2023 Trustees Report intermediate alternative assumptions (alt II), the range from 2023 Trustees Report low-cost (alt I) and 2023 Trustees Report high-cost assumptions (alt III), the median and bounds of the 95-percent frequency interval for 5,000 runs without parameter uncertainty, and the median and bounds of the 95-percent frequency interval for 5,000 runs with parameter uncertainty. For the mortality subsection, the increase in period life expectancy is shown rather than the 50-year and 75-year averages.

Unless otherwise noted, the average value is computed as an arithmetic average. When the geometric average is specified, the values are transformed by adding one, averaged geometrically, and reverse transformed by subtracting one. This method is analogous to computing an average effective annual rate for compound interest (Kellison 1991).

The following subsections contain figures displaying values of the key variables throughout their entire historical and projection periods. For most variables, the expected future annual and cumulative average values from the equations presented in chapter III are shown along with the 95-percent frequency intervals for 5,000 runs without parameter uncertainty and 5,000 runs with parameter uncertainty. The first figure displays the annual results and the second figure displays the cumulative average results. For the mortality subsection, the 95-percent frequency interval is shown for the annual values only. Note that the effects of the COVID-19 pandemic and the ensuing recession/recovery are evident through the first few years of the 75-year projection.

[23] A final 50-year average is presented because most of the variables in this section reach an assumed ultimate value prior to the end of the 25th projection year. Typically, this assumed ultimate value is a constant for that variable. A notable exception is the mortality assumption in which death rates do not reach a constant level because they are derived from a multiple decrement model (where causes of death compete). For mortality, *increases* in life expectancies over the 75-year and final 50 years of the projection period, rather than *average levels*, are presented.

1. Fertility

Table IV.1 lists the results of the total fertility rate equation presented in chapter III, and figure IV.1 displays a graphical representation of these results. The final 50-year average 95-percent frequency interval without parameter uncertainty is approximately 0.61 children per woman wider than the range defined by alternatives I and III. With parameter uncertainty, the 95-percent frequency interval is approximately 0.85 children per woman wider than the range defined by alternatives I and III. Note that stochastic results are approximately symmetrical around the median even though that is not true for alternative I and III relative to alternative II.

Table IV.1 – Total Fertility Rate:
Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	2.00	1.97	2.00
	Alt I and III Interval	[1.70, 2.20]	[1.68, 2.16]	[1.70, 2.20]
Stochastic Model – Without Parameter Uncertainty	Median	2.00	1.96	2.00
	95% Frequency Interval	[0.88, 3.08]	[1.53, 2.37]	[1.45, 2.56]
Stochastic Model – With Parameter Uncertainty	Median	2.00	1.97	2.00
	95% Frequency Interval	[0.83, 3.15]	[1.40, 2.52]	[1.32, 2.67]

Figure IV.1 – Total Fertility Rate: Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III; stochastic intervals are the 95-percent frequency intervals]

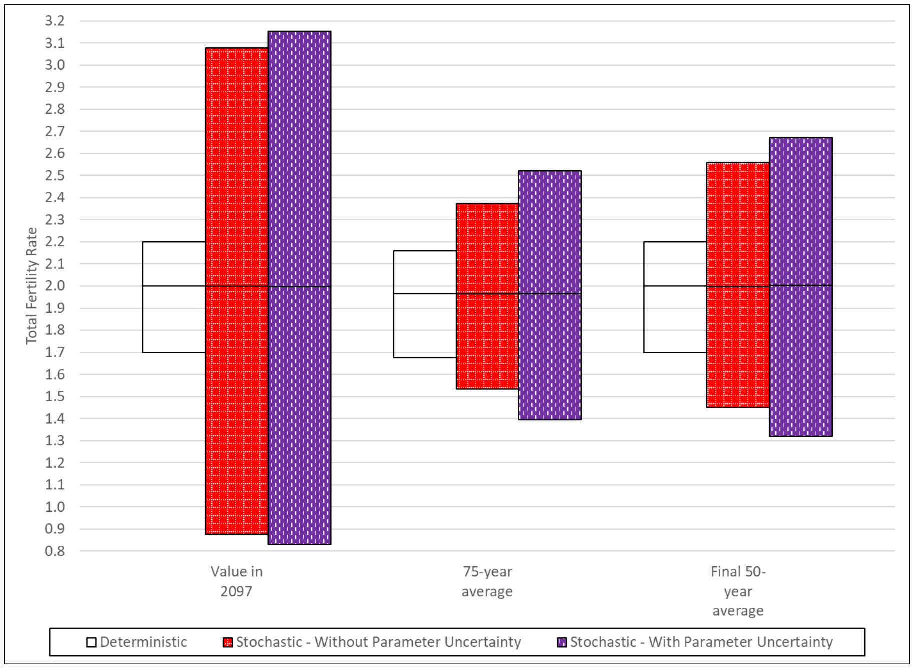


Figure IV.2a displays the annual total fertility rate throughout the entire historical and projection periods. Figure IV.2b displays the annual total fertility rate throughout the historical period and the cumulative average total fertility rate in the projection period. The annual values indicate that the average standard deviation of the final 50 years of the projected total fertility rate is about 0.56 children per woman without parameter uncertainty and 0.60 children per woman with parameter uncertainty.[24]

[24] Without parameter uncertainty, the width of the 95-percent frequency interval for the annual values for the final 50 years averages approximately 2.20 children per woman. With parameter uncertainty, the width of the 95-percent frequency interval for the annual values for the final 50 years averages approximately 2.33 children per woman. For a normal distribution, this width represents about four standard deviations.

Figure IV.2a – Total Fertility Rate Annual Values, Calendar Years 1917-2097

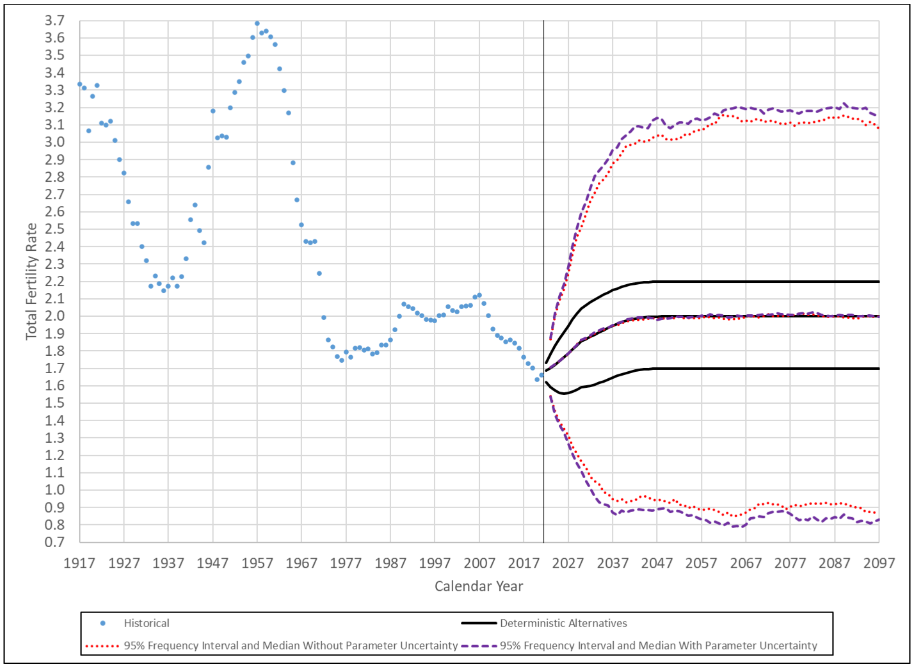
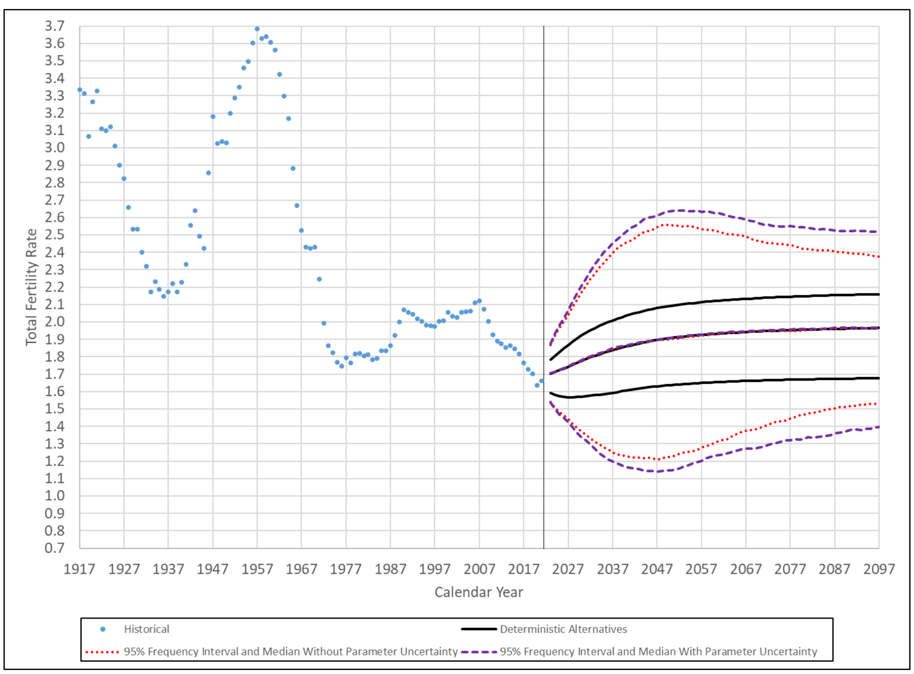


Figure IV.2b – Total Fertility Rate Cumulative Average Values, Calendar Years 1917-2097



2. Mortality

In order to simplify the presentation, mortality results are given here as male and female period life expectancies at birth and at age 65, separately. Thus, the number of age-sex groups is reduced from 42 to just four.

a. Period Life Expectancies at Birth

Tables IV.2 and IV.3 list the male and female period life expectancies at birth, resulting from the mortality equations presented in chapter III. The increases shown are calculated as the difference between the values in 2097 and 2023 or the difference between the values in 2097 and 2048. Without parameter uncertainty, the 95-percent frequency interval in 2097 is slightly lower than the lower bound of the alternative I and III range, and higher than the upper bound of the alternative I and III range. With parameter uncertainty, the 95-percent frequency interval expands on both sides.

Table IV.2 – Male Period Life Expectancies and Increase in Life Expectancies at Birth: Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year increase	Final 50-year increase
Deterministic Model	Alt II	83.41	7.35	4.52
	Alt I and III Interval	[78.68, 88.32]	[2.93, 11.89]	[1.83, 6.81]
Stochastic Model – Without Parameter Uncertainty	Median	84.32	8.26	4.97
	95% Frequency Interval	[77.91, 89.96]	[2.86, 12.84]	[3.01, 7.14]
Stochastic Model – With Parameter Uncertainty	Median	84.42	8.35	5.10
	95% Frequency Interval	[74.08, 91.67]	[-0.98, 14.54]	[-0.35, 8.51]

Table IV.3 – Female Period Life Expectancies and Increase in Life Expectancies at Birth: Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year increase	Final 50-year increase
Deterministic Model	Alt II	87.25	6.07	3.66
	Alt I and III Interval	[83.42, 91.11]	[2.49, 9.62]	[1.52, 5.39]
Stochastic Model – Without Parameter Uncertainty	Median	87.91	6.72	4.07
	95% Frequency Interval	[83.16, 93.40]	[2.73, 11.43]	[2.27, 6.59]
Stochastic Model – With Parameter Uncertainty	Median	87.92	6.73	4.06
	95% Frequency Interval	[81.05, 95.13]	[0.62, 13.15]	[0.46, 8.05]

Graphical representations of the data in table IV.3 are shown in figure IV.3 through figure IV.5.

Figure IV.3 – Period Life Expectancies at Birth in 2097: Selected Values and Intervals Under the Deterministic and Stochastic Models

[Deterministic interval is defined by alternatives I and III; stochastic intervals are the 95-percent frequency intervals]

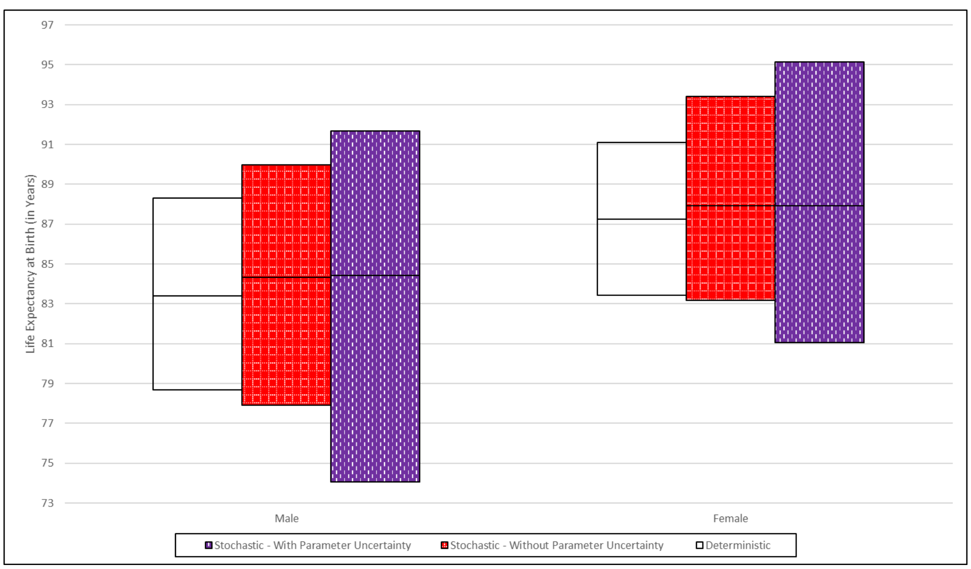


Figure IV.4 – 75-year Increases in Period Life Expectancies at Birth in 2097: Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III; stochastic intervals are the 95-percent frequency intervals]

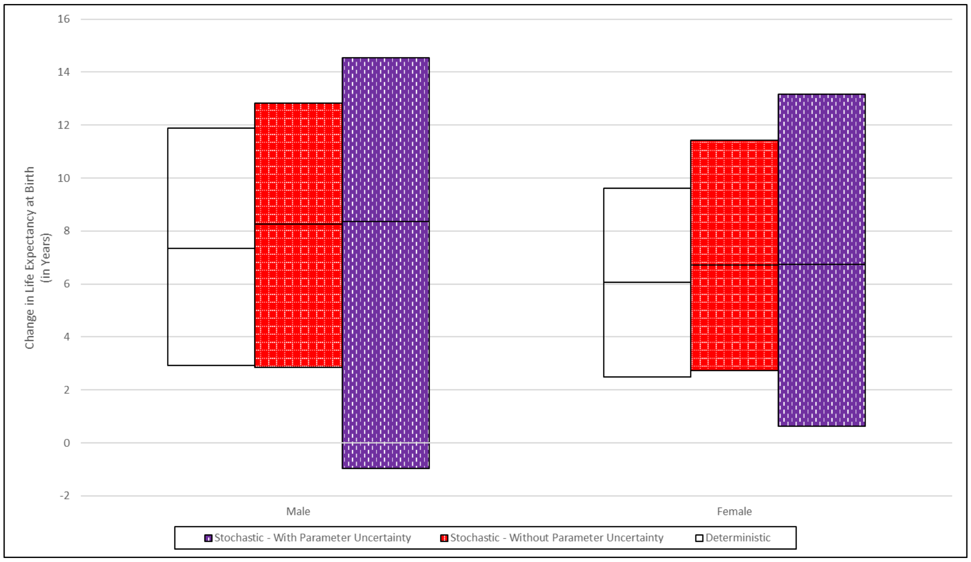
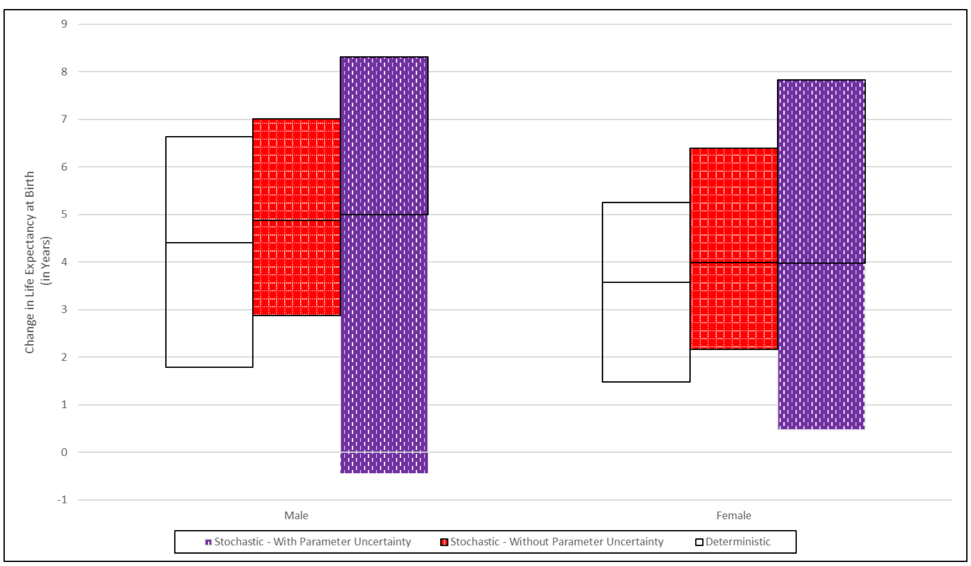


Figure IV.5 – Final 50-year Increases in Period Life Expectancies at Birth in 2097: Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III; stochastic intervals are the 95-percent frequency intervals]



Figures IV.6 and IV.7 display the male and female period life expectancies at birth, throughout the entire historical and projection periods.

Figure IV.6 – Male Period Life Expectancies at Birth, Calendar Years 1900-2097

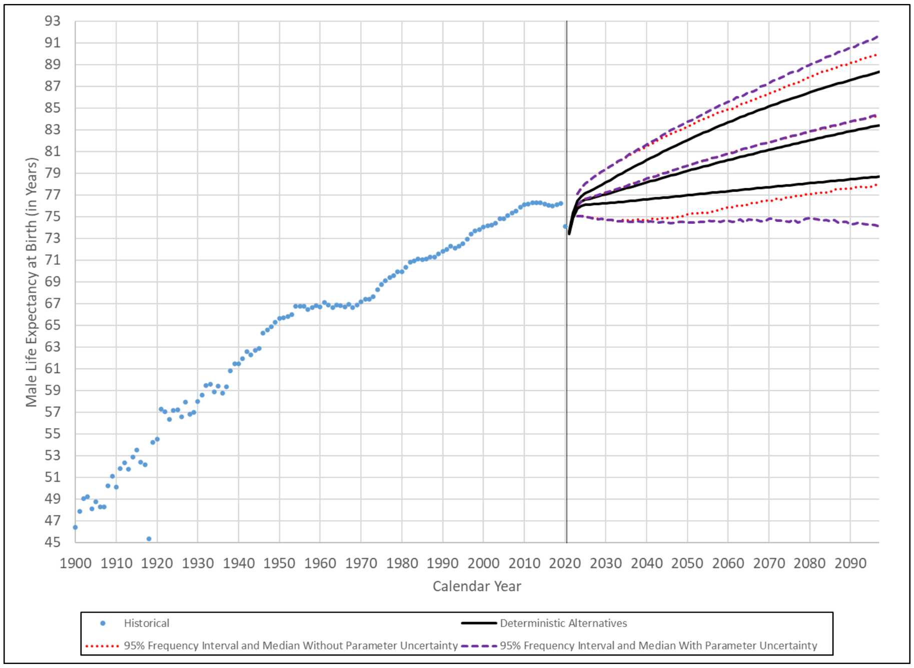
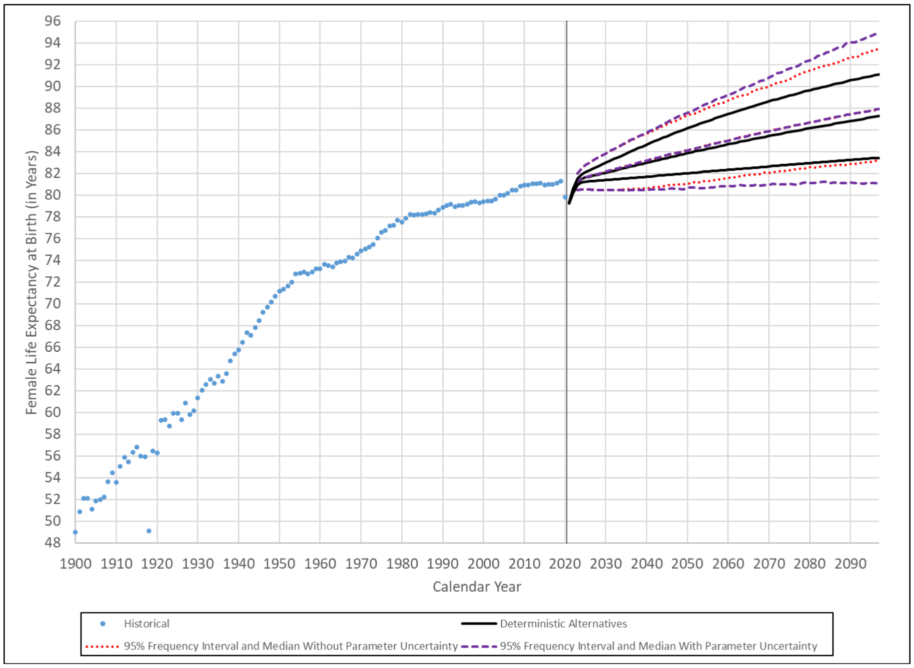


Figure IV.7 – Female Period Life Expectancies at Birth, Calendar Years 1900-2097



b. Period Life Expectancies at Age 65

Tables IV.4 and IV.5 show the period life expectancies at age 65 for men and women, respectively, resulting from the mortality equations presented in chapter III. The values shown are analogous to those shown in the previous subsection. As with the results for life expectancy at birth, the bounds of the 95-percent frequency intervals are farther from the upper bound of the alternative I and III range than from the lower bound. With parameter uncertainty, these bounds of the 95-percent frequency interval extend even farther, particularly at the upper end.

Table IV.4 – Male Period Life Expectancies and Increase in Life Expectancies at Age 65: Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year increase	Final 50-year increase
Deterministic Model	Alt II	22.56	4.46	2.72
	Alt I and III Interval	[19.71, 25.74]	[1.78, 7.42]	[1.07, 4.33]
Stochastic Model – Without Parameter Uncertainty	Median	22.95	4.84	2.96
	95% Frequency Interval	[19.24, 27.75]	[1.67, 9.07]	[1.31, 5.40]
Stochastic Model – With Parameter Uncertainty	Median	22.97	4.86	3.01
	95% Frequency Interval	[18.37, 28.89]	[0.80, 10.22]	[0.56, 6.43]

Table IV.5 – Female Period Life Expectancies and Increase in Life Expectancies at Age 65: Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year increase	Final 50-year increase
Deterministic Model	Alt II	24.68	3.98	2.41
	Alt I and III Interval	[22.14, 27.46]	[1.61, 6.56]	[0.95, 3.79]
Stochastic Model – Without Parameter Uncertainty	Median	25.16	4.46	2.71
	95% Frequency Interval	[21.22, 30.23]	[1.09, 8.92]	[0.97, 5.20]
Stochastic Model – With Parameter Uncertainty	Median	25.16	4.46	2.74
	95% Frequency Interval	[20.39, 31.75]	[0.25, 10.43]	[0.25, 6.60]

Graphical representations of the data in table IV.5 are shown in figure IV.8 through figure IV.10.

Figure IV.8 – Period Life Expectancies at Age 65 in 2097:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]

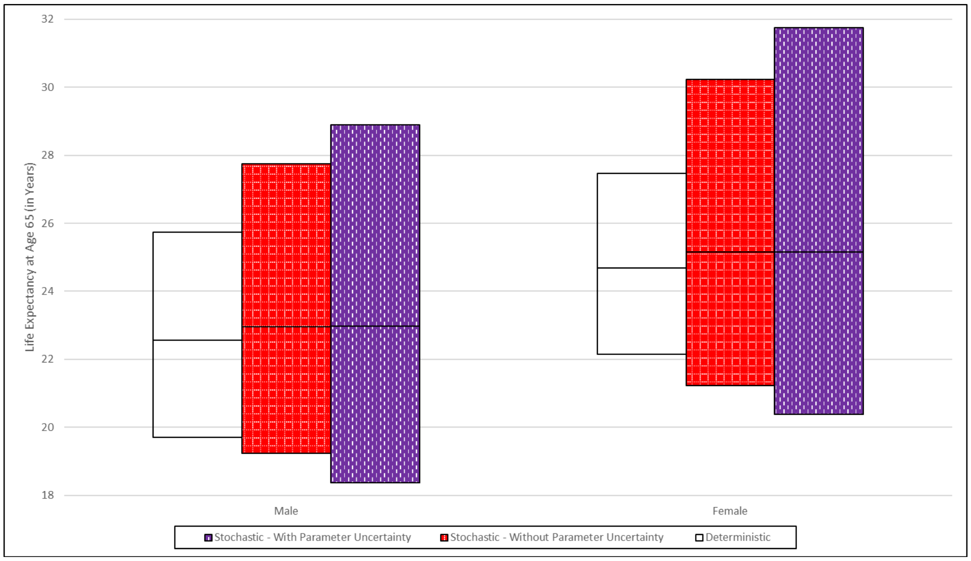


Figure IV.9 – 75-year Increases in Period Life Expectancies at Age 65 in 2097:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]

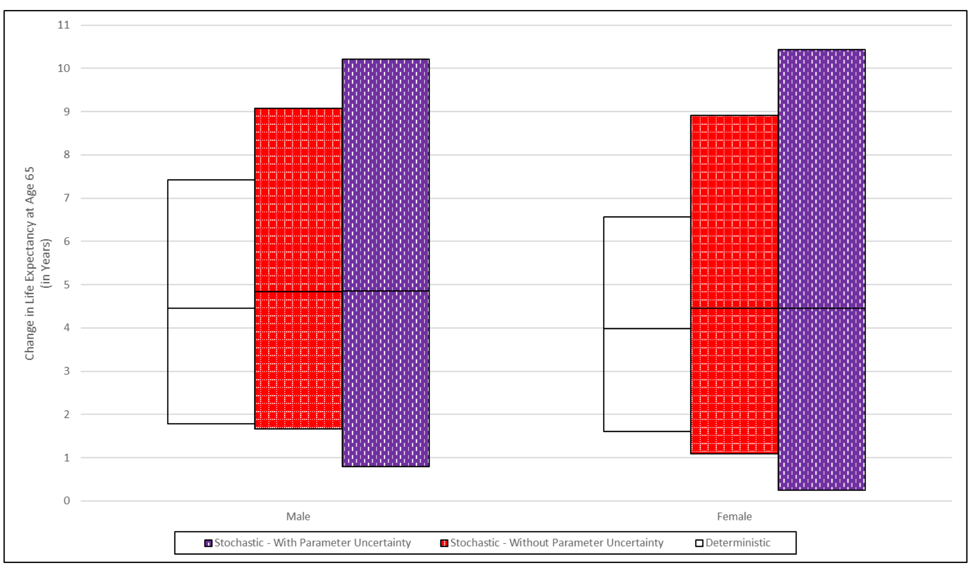
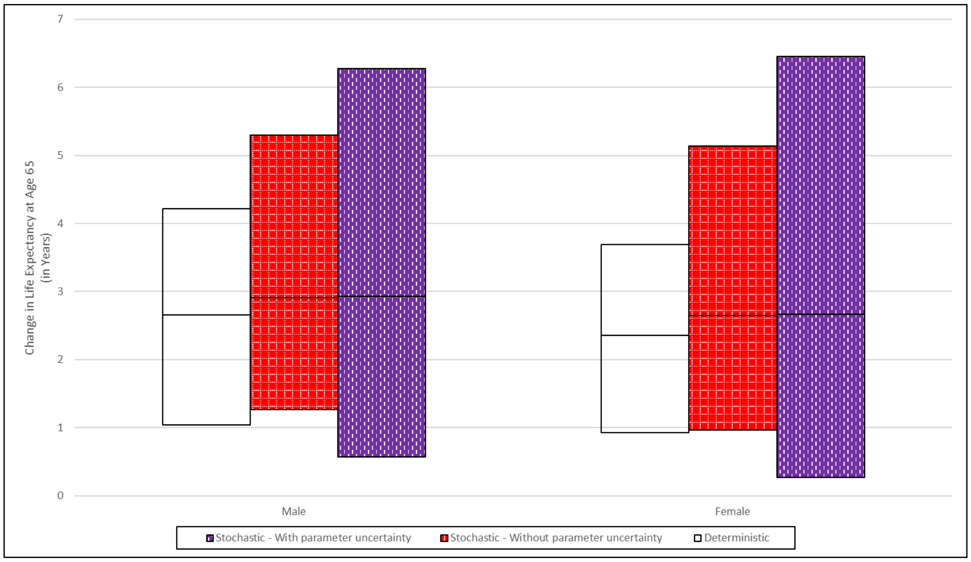


Figure IV.10 – Final 50-year Increases in Period Life Expectancies at Age 65 in 2097: Selected Values and Intervals Under the Deterministic and Stochastic Models

[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]



Figures IV.11 and IV.12 display the period life expectancies at age 65 for men and women, respectively, throughout the entire historical and projection periods.

Figure IV.11 – Male Period Life Expectancies at Age 65, Calendar Years 1900-2097

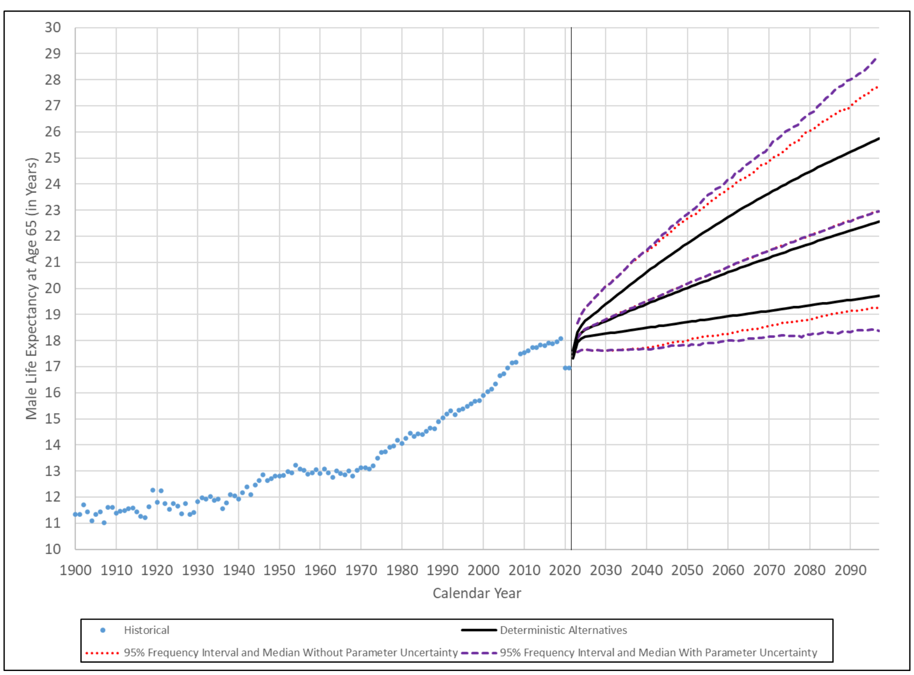
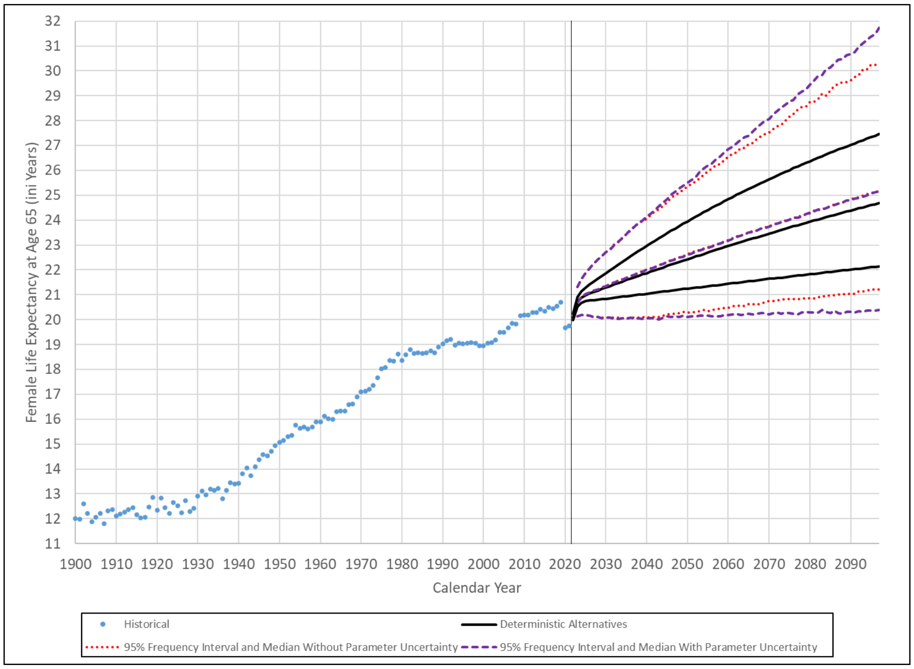


Figure IV.12 – Female Period Life Expectancies at Age 65, Calendar Years 1900-2097



3. Immigration

a. Lawful Permanent Resident (LPR) New Arrival Immigration

Table IV.6 displays the results (in thousands) of the LPR immigration equation presented in chapter III, and figure IV.13 displays a graphical representation of these results. For the final 50-year average, the lower bound of the 95-percent frequency interval is about 11,000 persons higher than the alternative III level, and the upper bound is about 13,000 persons lower than the alternative I level without parameter uncertainty. With parameter uncertainty, the lower bound of the 95-percent frequency interval is about 54,000 persons lower than the alternative III level, and the upper bound is about 56,000 persons higher than the alternative I level.

Table IV.6 – LPR New Arrival Immigration (in thousands): Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	600	607	600
	Alt I and III Interval	[500, 700]	[507, 707]	[500, 700]
Stochastic Model – Without Parameter Uncertainty	Median	601	606	600
	95% Frequency Interval	[356, 850]	[533, 681]	[511, 687]
Stochastic Model – With Parameter Uncertainty	Median	602	608	603
	95% Frequency Interval	[329, 879]	[466, 748]	[446, 756]

Figure IV.13 – LPR New Arrival Immigration: Selected Values and Intervals Under the Deterministic and Stochastic Models

[Deterministic interval is defined by alternatives I and III; stochastic intervals are the 95-percent frequency intervals]

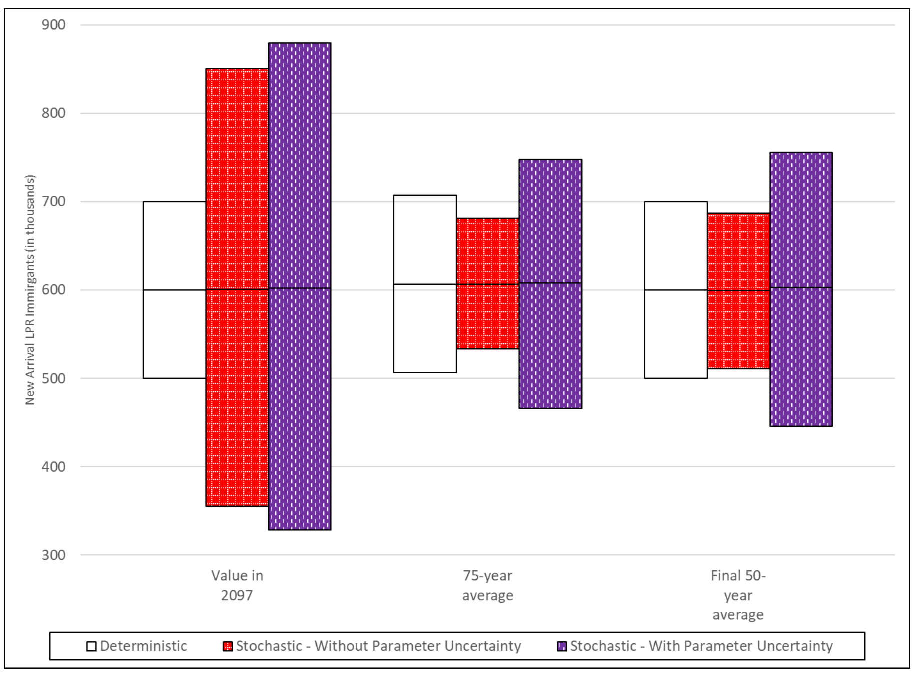


Figure IV.14a displays the annual new LPR arrivals throughout the entire historical and projection periods. Figure IV.14b displays the annual new LPR arrivals throughout the historical period and the cumulative average new LPR arrivals in the projection period. The annual values indicate that the average standard deviation of the final 50 years of the projected annual new LPR arrivals is about 128,000 without parameter uncertainty and 143,000 with parameter uncertainty.[25]

Figure IV.14a – LPR New Arrival Immigration Annual Values, Calendar Years 1991-2097

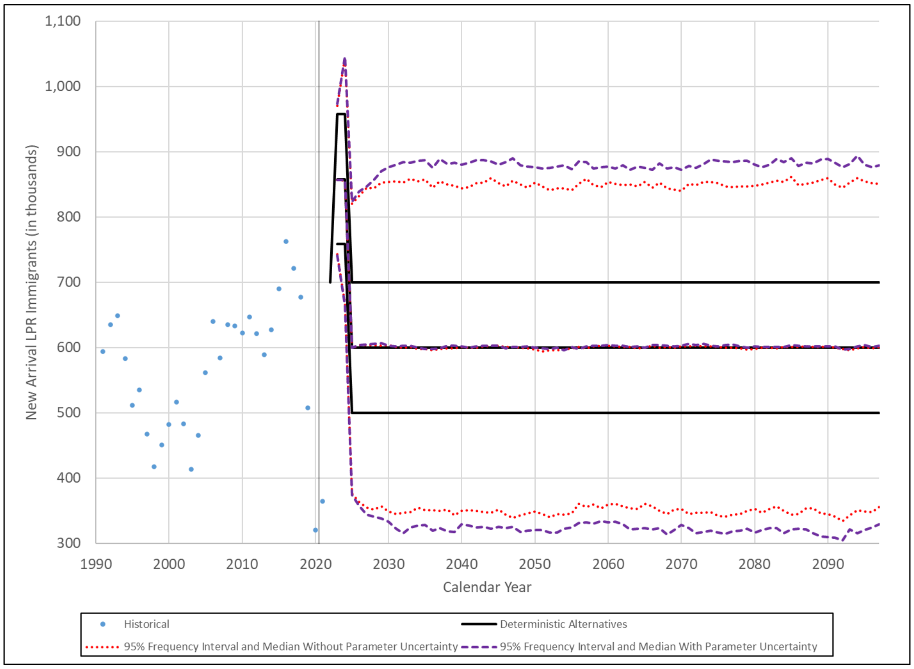
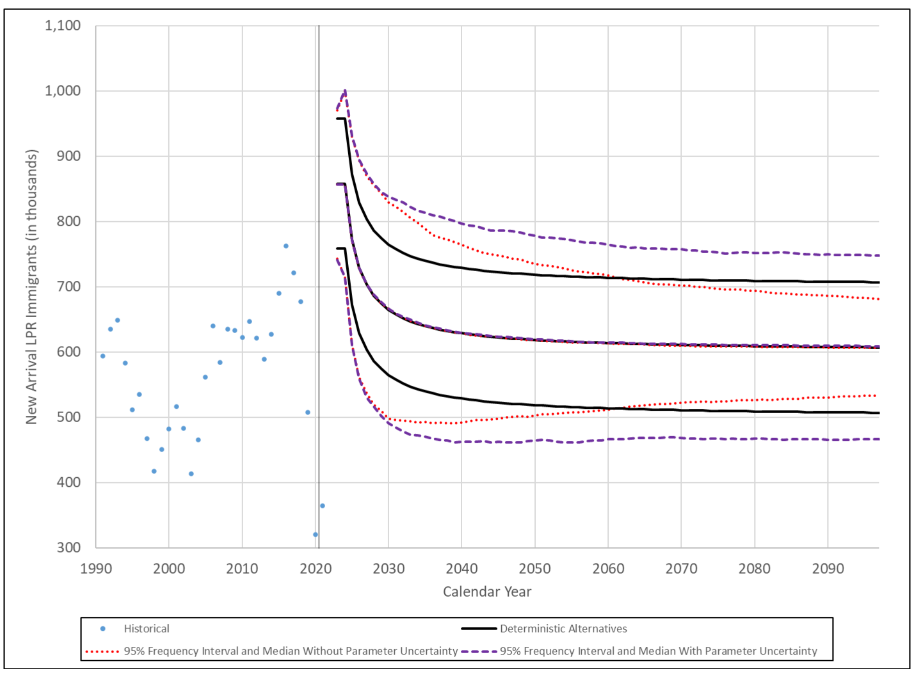


Figure IV.14b – LPR New Arrival Immigration Cumulative Average Values, Calendar Years 1991-2097



[25] Without parameter uncertainty, the width of the 95-percent frequency interval for the annual values for the final 50 years averages approximately 500,000. With parameter uncertainty, the width of the 95-percent frequency interval for the annual values for the final 50 years averages approximately 559,000. For a normal distribution, this width represents about four standard deviations.

b. Adjustments of Status

As described in chapter III, "adjustments of status" are those persons within the United States who adjust from other-than-LPR status to LPR status. The transfer rate is the percentage of the beginning-of-year other-than-LPR population that adjusts status to become LPRs. Table IV.7 displays the results (in thousands) of the transfer rate equation (transformed back to actual numbers of adjustments of status) presented in chapter III, and figure IV.15 displays a graphical representation of these results. Without parameter uncertainty, the lower bound of the final 50-year average 95-percent frequency interval is about 33,000 persons higher than the alternative III value, and the upper bound is about 1,000 persons higher than the alternative I value. With parameter uncertainty, the lower bound of the 95-percent frequency interval is about 18,000 persons lower than the alternative III value, and the upper bound is about 63,000 persons higher than the alternative I value.

Table IV.7 – Adjustments of Status (in thousands):
Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	450	450	450
	Alt I and III Interval	[350,550]	[350, 550]	[350, 550]
Stochastic Model – Without Parameter Uncertainty	Median	445	460	459
	95% Frequency Interval	[266, 749]	[396, 532]	[383, 551]
Stochastic Model – With Parameter Uncertainty	Median	443	459	458
	95% Frequency Interval	[242, 794]	[351, 586]	[332, 613]

Figure IV.15 – Adjustments of Status:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]

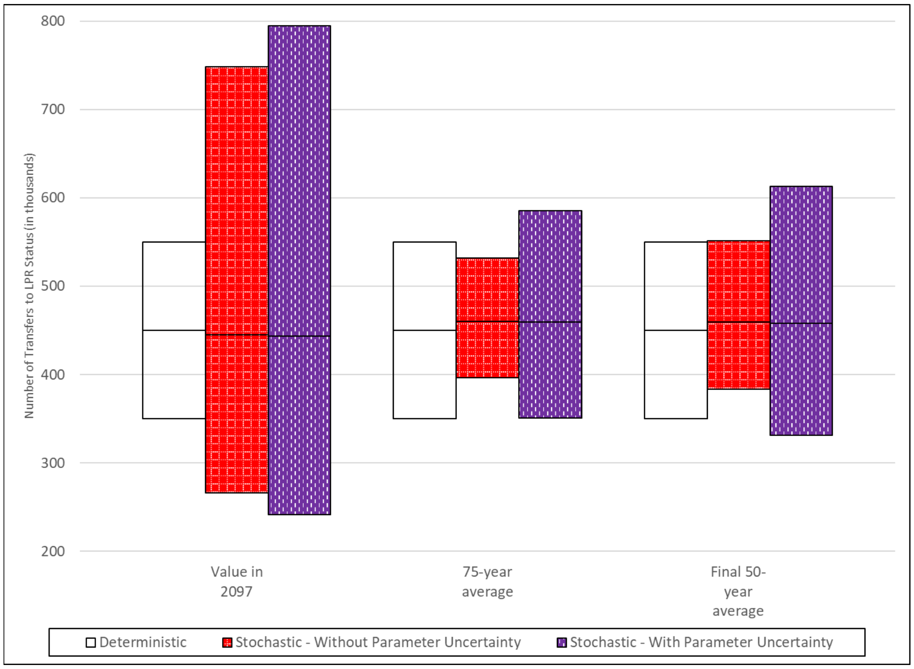


Figure IV.16a displays the annual transfers throughout the entire historical and projection periods. Figure IV.16b displays the annual transfers throughout the historical period and the cumulative average transfers in the projection period. Because transfer *rates* are applied to the other-than-LPR stock, the frequency intervals of final transfer values are not quite symmetrical around the median value. The annual values indicate that the average standard deviation of the final 50 years of the projected annual adjustments of status is about 124,000 without parameter uncertainty and 135,000 with parameter uncertainty.[26]

[26] Without parameter uncertainty, the width of the 95-percent frequency interval for the annual values for the final 50 years averages approximately 484,000. With parameter uncertainty, the width of the 95-percent frequency interval for the annual values for the final 50 years averages approximately 527,000. For a normal distribution, this width represents about four standard deviations.

Figure IV.16a – Adjustments of Status Annual Values, Calendar Years 1978-2097

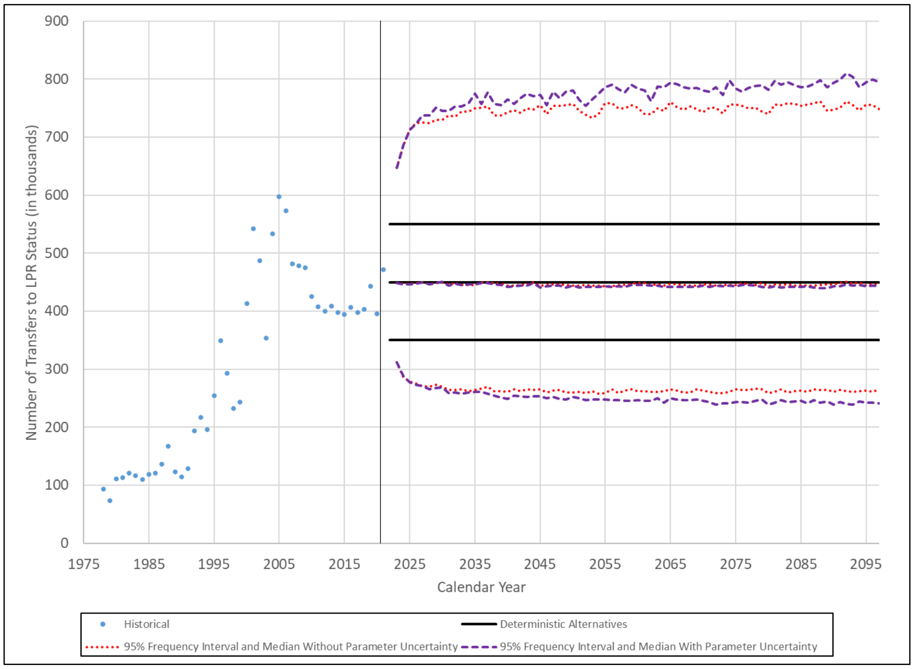
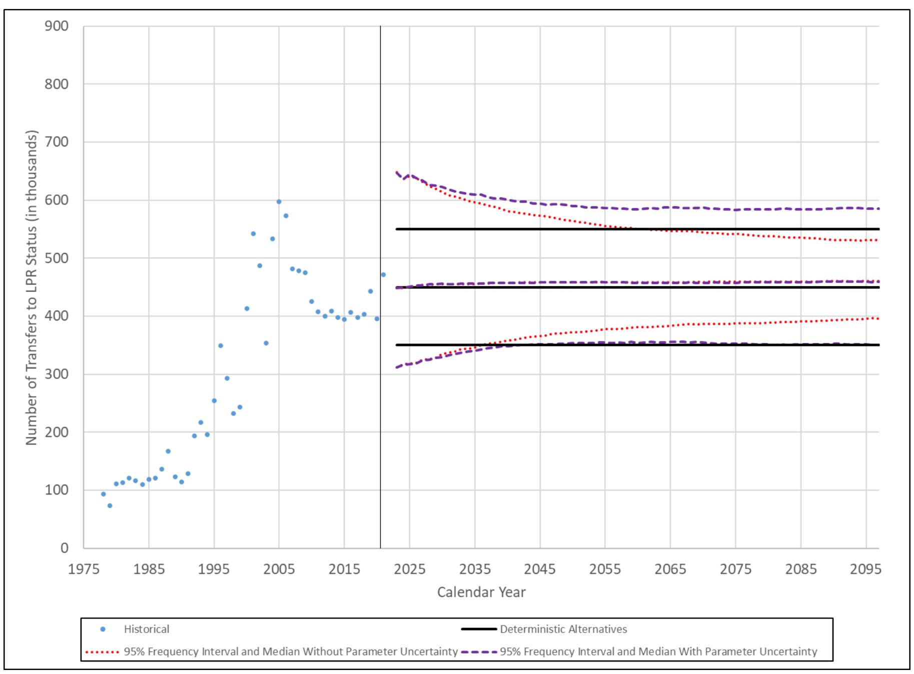


Figure IV.16b – Adjustments of Status Cumulative Average Values, Calendar Years 1978-2097



c. Legal Emigration

Table IV.8 displays the results of the legal emigration rate of beginning of year LPR plus citizen population equation presented in chapter III, and figure IV.17 displays a graphical representation of these results. The final 50-year average 95-percent frequency interval is about 0.015 percent wider than the alternative I and III range without parameter uncertainty. With parameter uncertainty, the final 50-year average 95-percent frequency interval is about 0.025 percent wider to the alternative I and III range.

Table IV.8 – Legal Emigration Rate:
Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	0.060	0.070	0.066
	Alt I and III Interval	[0.047, 0.074]	[0.061, 0.075]	[0.056, 0.074]
Stochastic Model – Without Parameter Uncertainty	Median	0.060	0.072	0.068
	95% Frequency Interval	[0.035, 0.100]	[0.060, 0.088]	[0.054, 0.086]
Stochastic Model – With Parameter Uncertainty	Median	0.059	0.073	0.068
	95% Frequency Interval	[0.034, 0.103]	[0.056, 0.094]	[0.050, 0.093]

Figure IV.17 – Legal Emigration Rate:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]

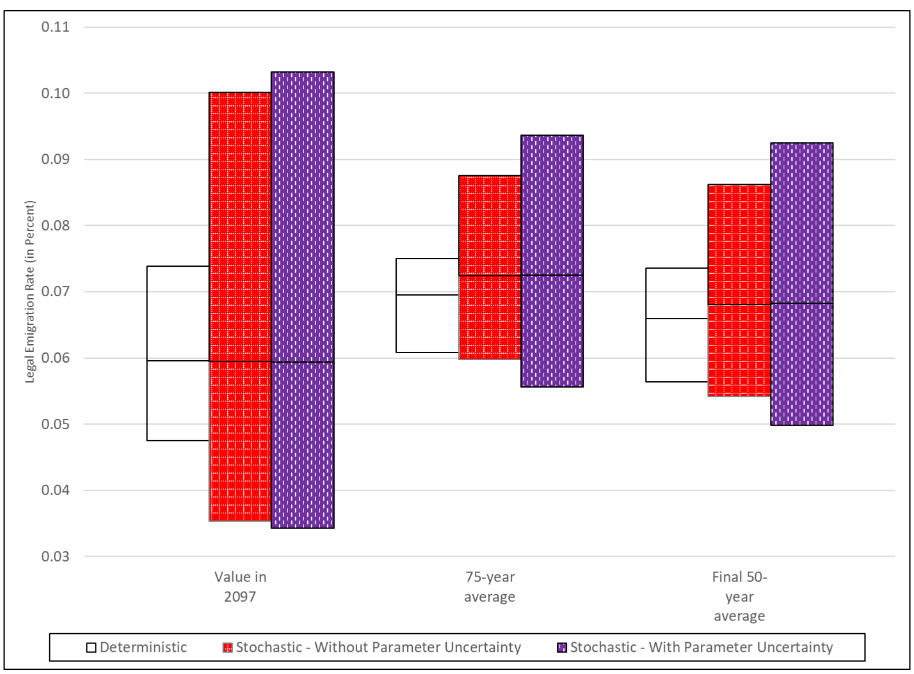
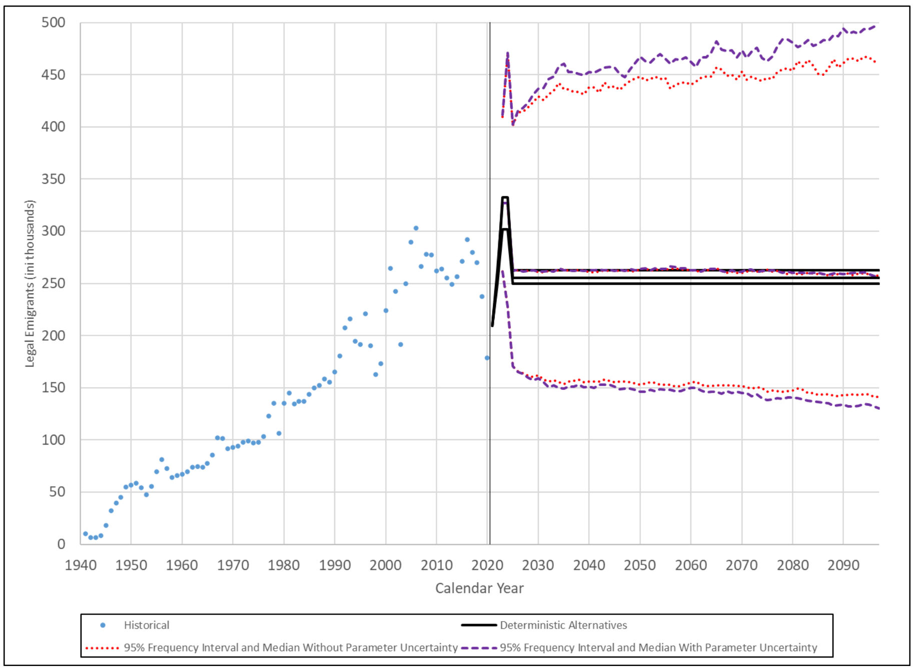


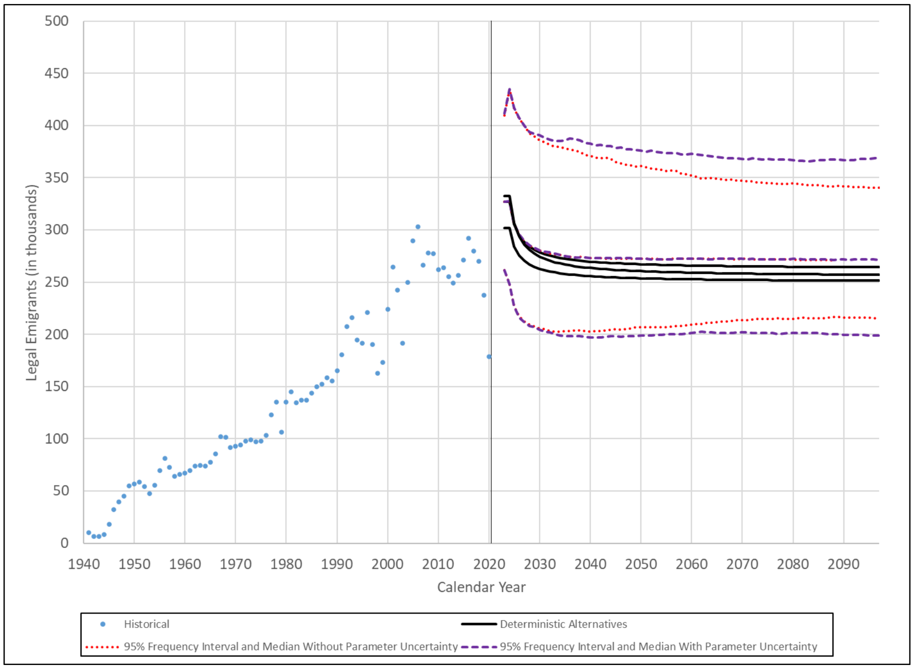
Figure IV.18a displays the annual legal emigration level resulting throughout the entire historical and projection periods. Figure IV.18b displays the annual legal emigration level resulting throughout the historical period and the cumulative average legal emigration level in the projection period. Because the original equation is based on *rates* that are applied to the LPR plus citizen stock, the frequency intervals of final legal emigration values are not quite symmetrical around the median value. The annual values indicate that the average standard deviation of the final 50 years of the projected annual legal emigration is about 76,000 without parameter uncertainty and 84,000 with parameter uncertainty.[27] Also noteworthy with figures IV.18a and IV.18b is the narrow range of the deterministic alternatives relative to the 95-percent frequency interval of the stochastic model.

Figure IV.18a – Legal Emigration Annual Values, Calendar Years 1941-2097



[27] Without parameter uncertainty, the width of the 95-percent frequency interval for the annual values for the final 50 years averages approximately 297,000. With parameter uncertainty, the width of the 95-percent frequency interval for the annual values for the final 50 years averages approximately 326,000. For a normal distribution, this width represents about four standard deviations.

Figure IV.18b – Legal Emigration Cumulative Average Values, Calendar Years 1941-2097



d. Other-than-LPR Immigration

Table IV.9 displays the results (in thousands) of the other-than-LPR immigration equation presented in chapter III, and figure IV.19 displays a graphical representation of these results. The final 50-year average 95-percent frequency interval is about 696,000 narrower than the alternative I and III range without parameter uncertainty. With parameter uncertainty, the final 50-year average 95-percent frequency interval is closer to the alternative I and III range but still somewhat narrower.

**Table IV.9 – Other-than-LPR Immigration (in thousands):
Selected Values and Intervals Under the Deterministic and Stochastic Models**

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	1350	1361	1350
	Alt I and III Interval	[850, 1850]	[861, 1861]	[850, 1850]
Stochastic Model – Without Parameter Uncertainty	Median	1363	1359	1349
	95% Frequency Interval	[738, 1958]	[1206, 1510]	[1168, 1537]
Stochastic Model – With Parameter Uncertainty	Median	1359	1360	1348
	95% Frequency Interval	[652, 2019]	[1020, 1698]	[971, 1719]

**Figure IV.19 – Other-than-LPR Immigration:
Selected Values and Intervals Under the Deterministic and Stochastic Models**
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]

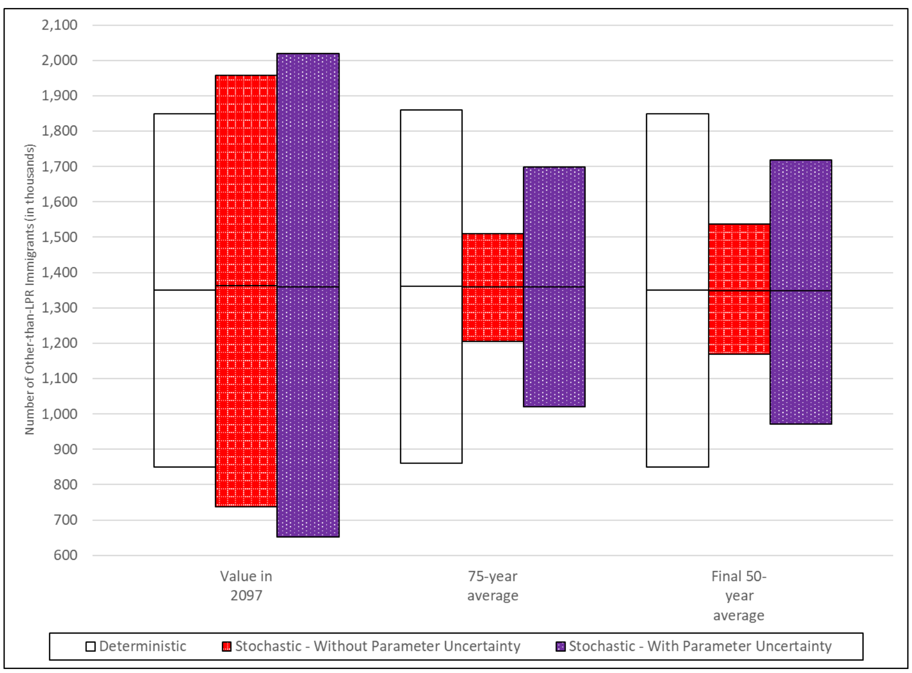


Figure IV.20a displays the annual other-than-LPR immigration level throughout the entire historical and projection periods. Figure IV.20b displays the annual other-than-LPR immigration level throughout the historical period and the cumulative average other-than-LPR immigration level in the projection period. The annual values indicate that the average standard deviation of the final 50 years of the projected annual other-than-LPR immigration is about 310,000 without parameter uncertainty and 350,000 with parameter uncertainty.[28]

Figure IV.20a – Other-than-LPR Immigration Annual Values, Calendar Years 1999-2097

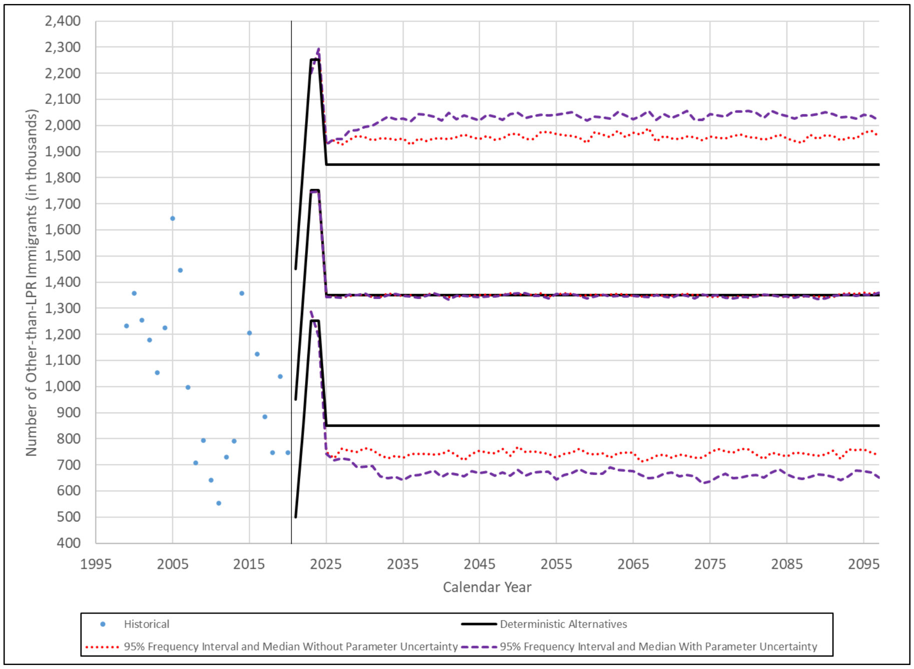
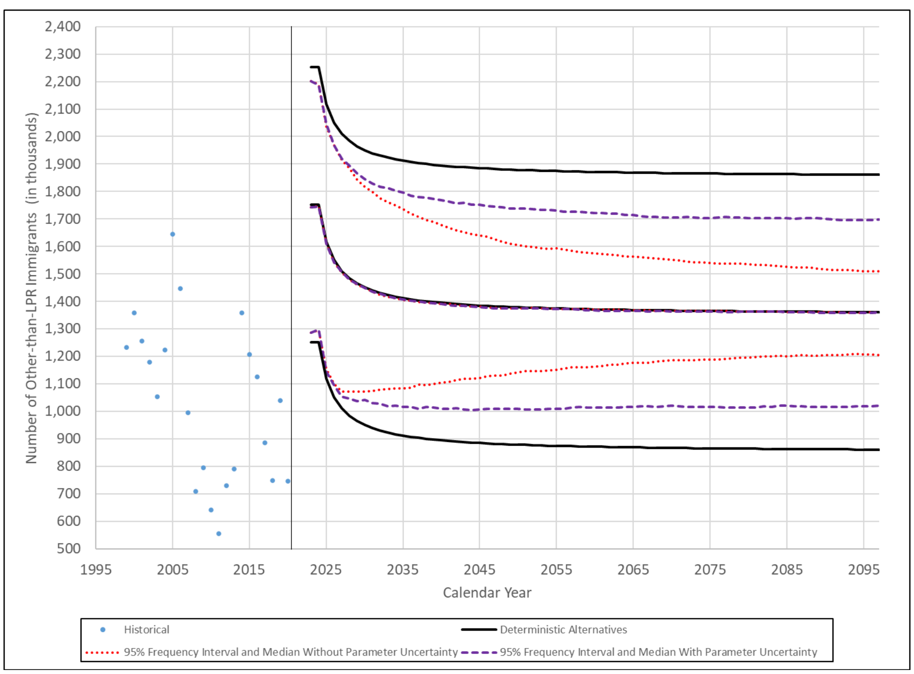


Figure IV.20b – Other-than LPR Immigration Cumulative Average Values, Calendar Years 1999-2097



[28] Without parameter uncertainty, the width of the 95-percent frequency interval for the annual values for the final 50 years averages approximately 1,212,000. With parameter uncertainty, the width of the 95-percent frequency interval averages approximately 1,372,000. For a normal distribution, this width represents about four standard deviations.

4. Unemployment Rate

Table IV.10 shows the results of the unemployment rate equation presented in chapter III, and figure IV.21 displays a graphical representation of these results. The final 50-year average 95-percent frequency interval (3.96 percent to 5.27 percent) is a bit narrower than the alternative I and III range (3.51 percent to 5.26 percent) without parameter uncertainty but shifted up slightly due to the log-odds transformation of the unemployment rate. With parameter uncertainty, the frequency interval is close in size to the alternative I and III range but also shifted upwards (3.75 percent to 5.53 percent).

Table IV.10 – Unemployment Rate (in percent):
Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	4.43	4.43	4.43
	Alt I and III Interval	[3.51, 5.26]	[3.50, 5.31]	[3.51, 5.26]
Stochastic Model – Without Parameter Uncertainty	Median	4.42	4.57	4.58
	95% Frequency Interval	[2.50, 7.65]	[4.06, 5.13]	[3.96, 5.27]
Stochastic Model – With Parameter Uncertainty	Median	4.42	4.56	4.57
	95% Frequency Interval	[2.47, 7.70]	[3.86, 5.37]	[3.75, 5.53]

Figure IV.21 – Unemployment Rate:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]

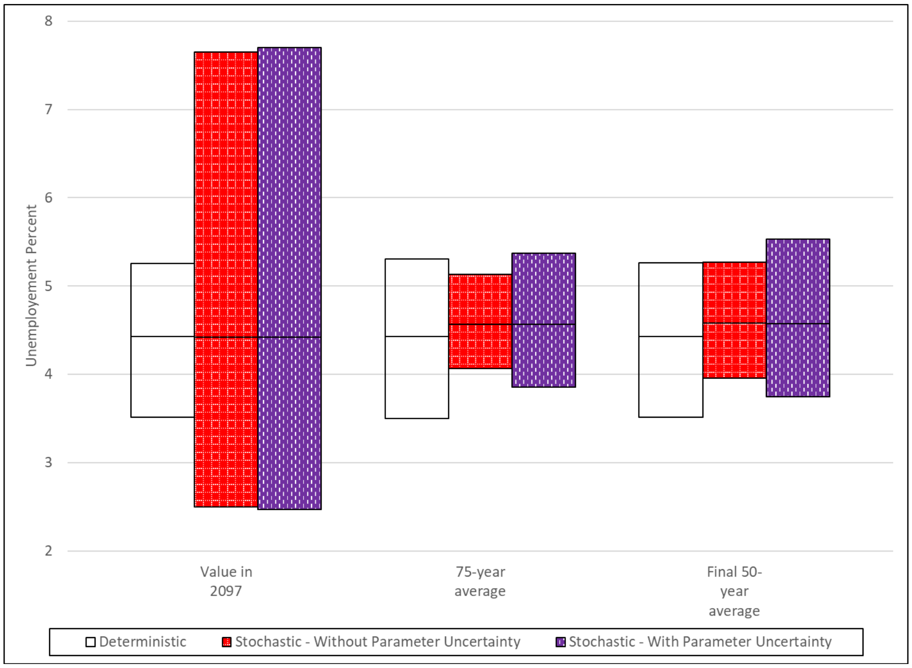


Figure IV.22a displays the annual unemployment rate throughout the entire historical and projection periods. Figure IV.22b displays the annual unemployment rate throughout the historical period and the cumulative average unemployment rate in the projection period. Due to the log-odds transformation of the unemployment rate variable, the frequency intervals are not symmetrical around the median value. This asymmetry is intended and desirable, as the typical values of the unemployment rate are much closer to its natural lower bound (zero) than to its natural upper bound (100 percent).

Figure IV.22a – Unemployment Rate Annual Values, Calendar Years 1961-2097

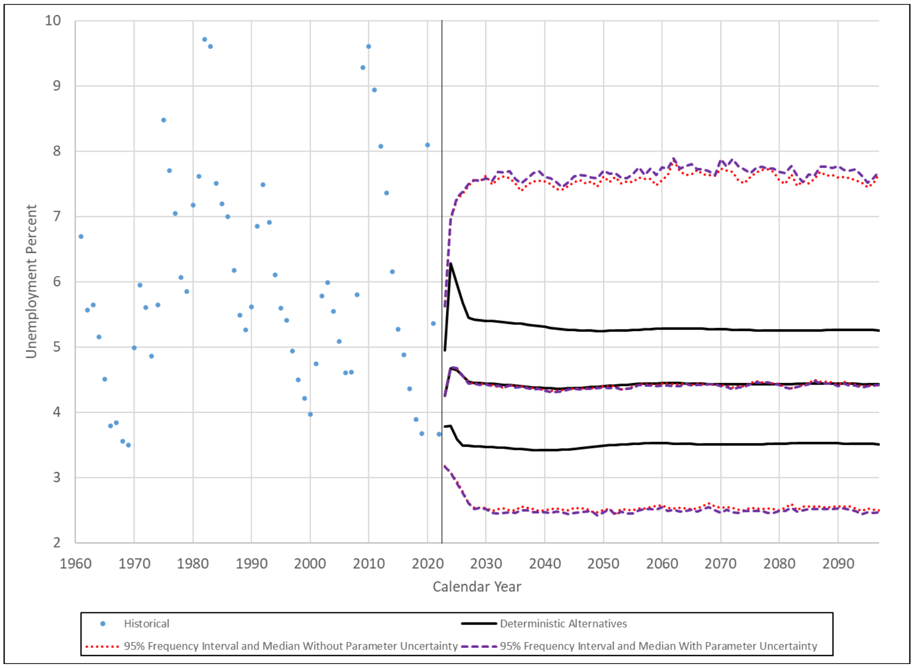
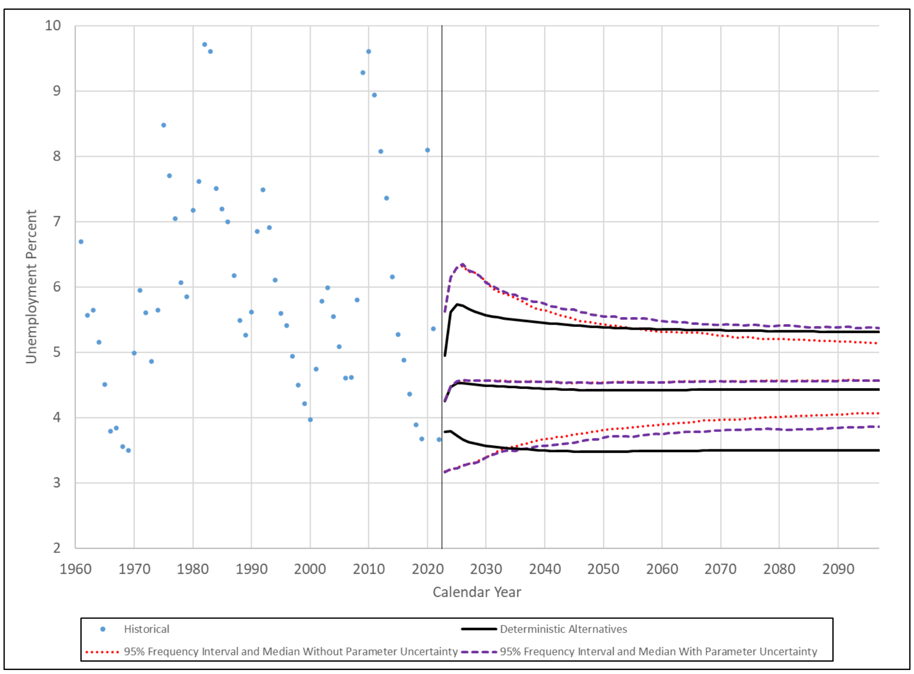


Figure IV.22b – Unemployment Rate Cumulative Average Values, Calendar Years 1961-2097



5. Inflation Rate

Table IV.11 displays the results of the CPI-W inflation rate equation presented in chapter III, and figure IV.23 displays a graphical representation of these results. The values presented in the table are growth rates expressed in percent, and the averages shown are geometric averages. Without parameter uncertainty, the 50-year geometric average 95-percent frequency interval is much wider than the alternative I and III range but shifted up slightly due to the logarithmic transformation of the inflation rate. With parameter uncertainty, this frequency interval is even wider, with the upward shift remaining.

Table IV.11 – Inflation Rate (in percent):
Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	2.40	2.42	2.40
	Alt I and III Interval	[1.80, 3.00]	[1.85, 3.01]	[1.80, 3.00]
Stochastic Model – Without Parameter Uncertainty	Median	2.39	2.69	2.67
	95% Frequency Interval	[-0.24, 7.70]	[1.47, 4.27]	[1.22, 4.60]
Stochastic Model – With Parameter Uncertainty	Median	2.34	2.70	2.67
	95% Frequency Interval	[-0.44, 8.66]	[0.89, 5.27]	[0.71, 5.56]

Figure IV.23 – Inflation Rate:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]

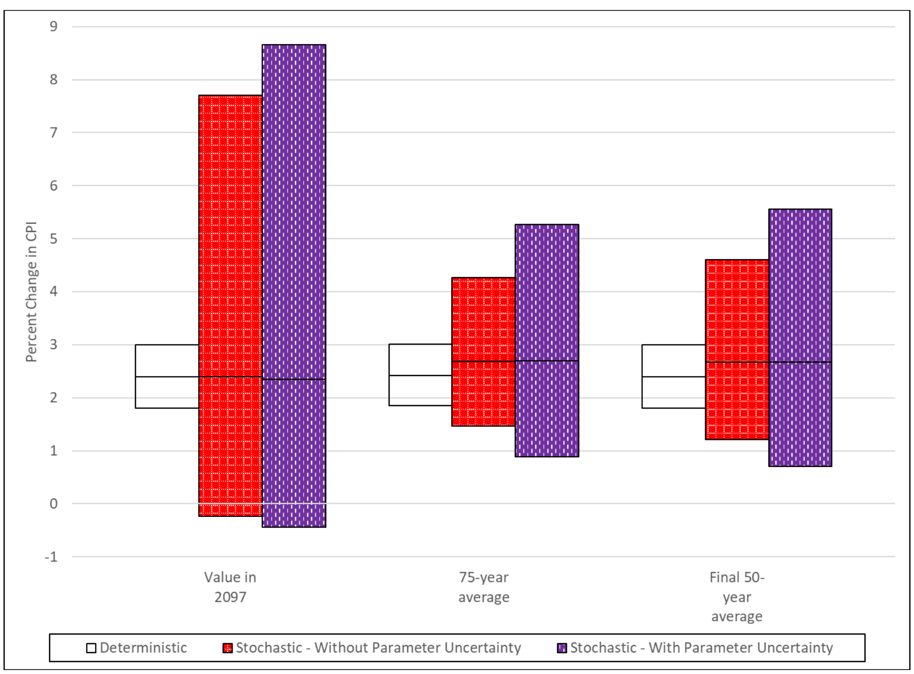


Figure IV.24a displays the annual inflation rate throughout the entire historical and projection periods. Figure IV.24b displays the annual inflation rate throughout the historical period and the cumulative average inflation rate in the projection period. The asymmetry in the frequency intervals around the median values is due to the logarithmic transformation that was applied to the inflation rate variable. This asymmetry is intended and desirable, as episodes of high inflation are empirically much more common than episodes of significant deflation.

Figure IV.24a – Inflation Rate Annual Values, Calendar Years 1961-2097

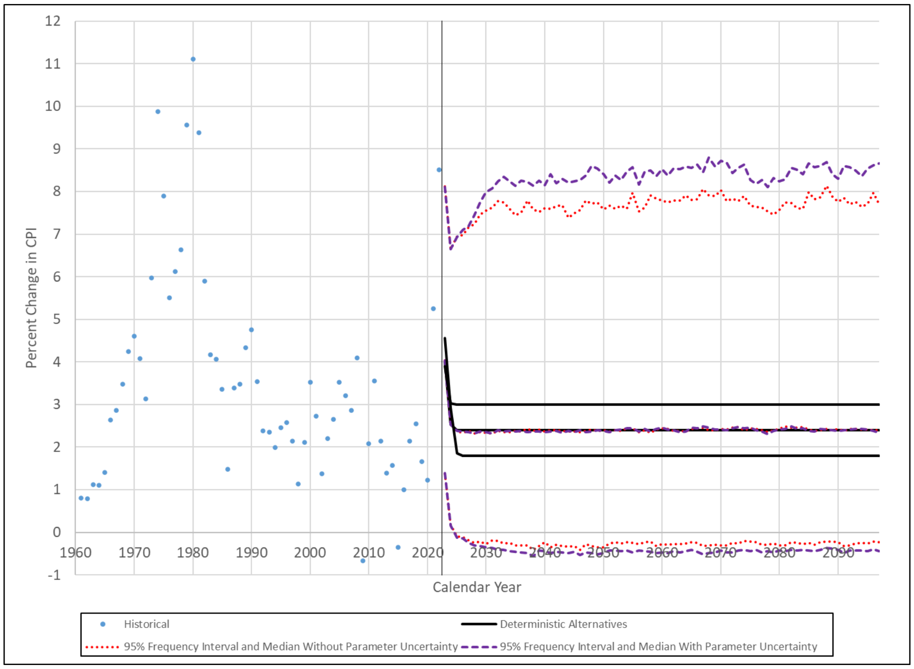
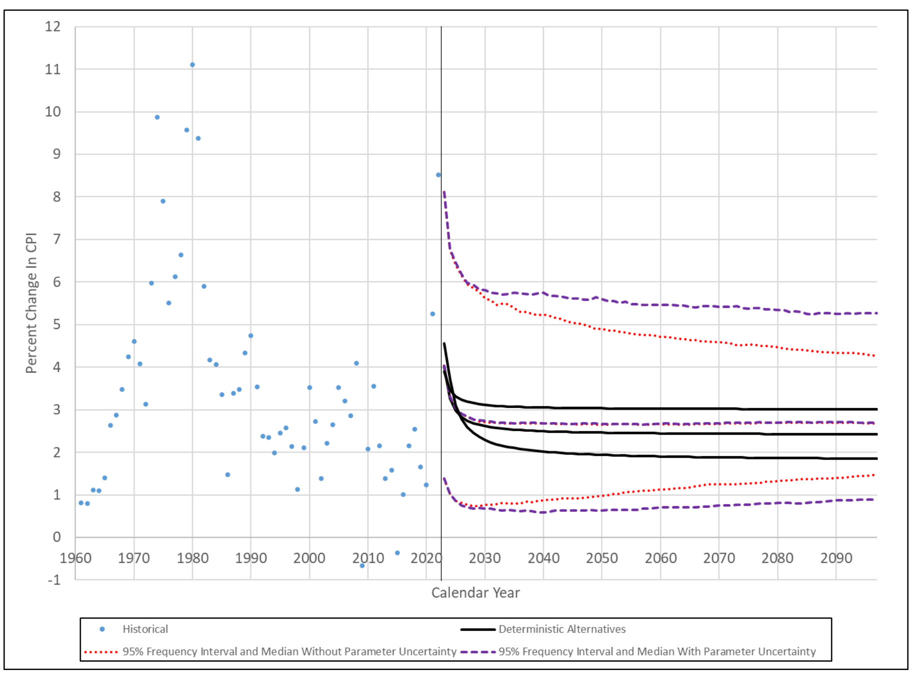


Figure IV.24b – Inflation Rate Cumulative Average Values, Calendar Years 1961-2097



6. Real Interest Rate

Table IV.12 shows the results of the real interest rate equation presented in chapter III, and figure IV.25 displays a graphical representation of these results. The values presented in the table are expressed in percent, and the averages shown are geometric averages. The 50-year geometric average 95-percent frequency interval is significantly wider than the alternative I and III range without parameter uncertainty. The frequency interval is even wider with parameter uncertainty.

Table IV.12 – Real Interest Rate (in percent):
Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	2.30	2.17	2.30
	Alt I and III Interval	[1.80, 2.80]	[1.69, 2.67]	[1.80, 2.80]
Stochastic Model – Without Parameter Uncertainty	Median	2.37	2.22	2.36
	95% Frequency Interval	[-2.20, 7.71]	[0.84, 3.82]	[0.63, 4.43]
Stochastic Model – With Parameter Uncertainty	Median	2.41	2.23	2.38
	95% Frequency Interval	[-2.19, 8.15]	[0.38, 4.76]	[0.28, 5.24]

Figure IV.25 – Real Interest Rate:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]

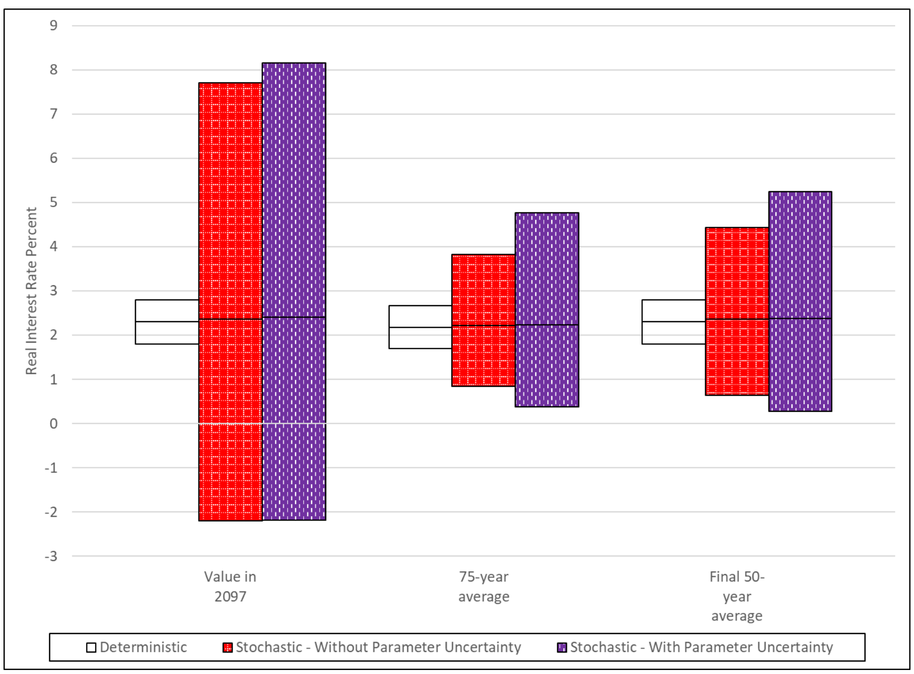


Figure IV.26a displays the real interest rate on new special issue securities throughout the entire historical and projection periods. Figure IV.26b displays the real interest rate on new special issue securities throughout the historical period and the cumulative average real interest rate on new special issue securities in the projection period.

Figure IV.26a – Real Interest Rate Annual Values, Calendar Years 1961-2097

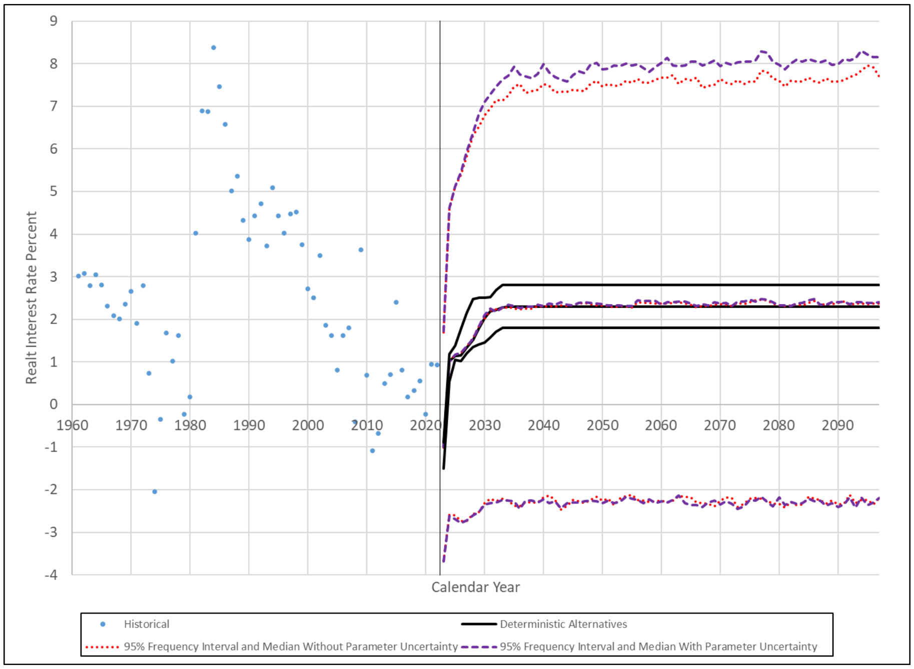
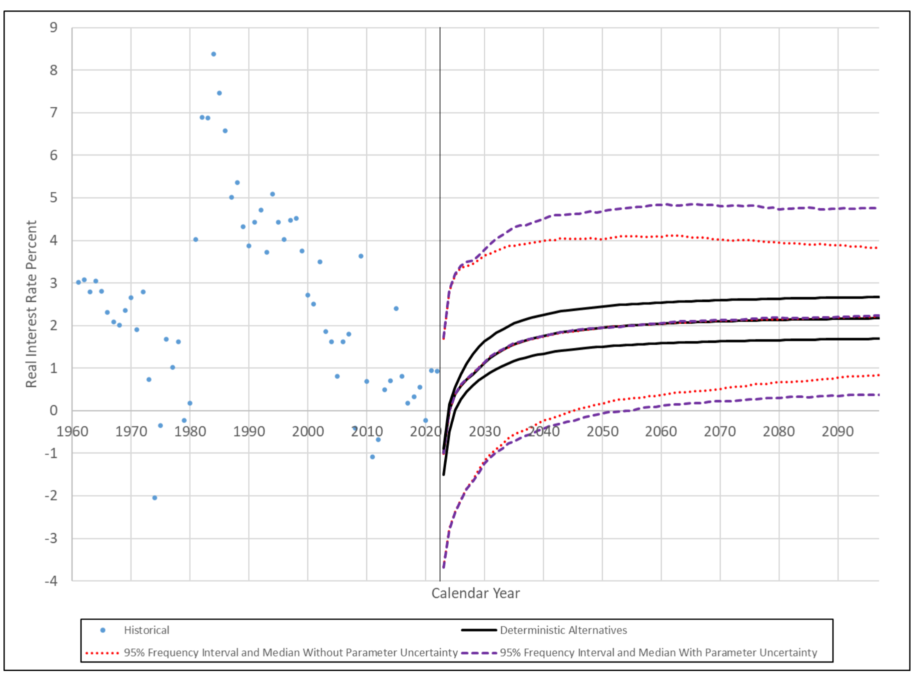


Figure IV.26b – Real Interest Rate Cumulative Average Values, Calendar Years 1961-2097



7. Real Average Covered Wage

Table IV.13 displays the results of the real average covered wage equation presented in chapter III, and figure IV.27 displays a graphical representation of these results. The values presented in the table are growth rates expressed in percent, and the averages shown are geometric averages. Without parameter uncertainty, the 50-year geometric average 95-percent frequency interval is slightly narrower than the alternative I and III range. With parameter uncertainty, the frequency interval is very close to the alternative I and III range.

Table IV.13 – Percentage Change in Real Average Covered Wage

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	1.12	1.17	1.12
	Alt I and III Interval	[0.53, 1.72]	[0.57, 1.80]	[0.52, 1.72]
Stochastic Model – Without Parameter Uncertainty	Median	1.11	1.16	1.12
	95% Frequency Interval	[-2.18, 4.41]	[0.74, 1.58]	[0.60, 1.65]
Stochastic Model – With Parameter Uncertainty	Median	1.09	1.17	1.12
	95% Frequency Interval	[-2.20, 4.42]	[0.60, 1.74]	[0.46, 1.80]

Figure IV.27 – Real Average Covered Wage:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]

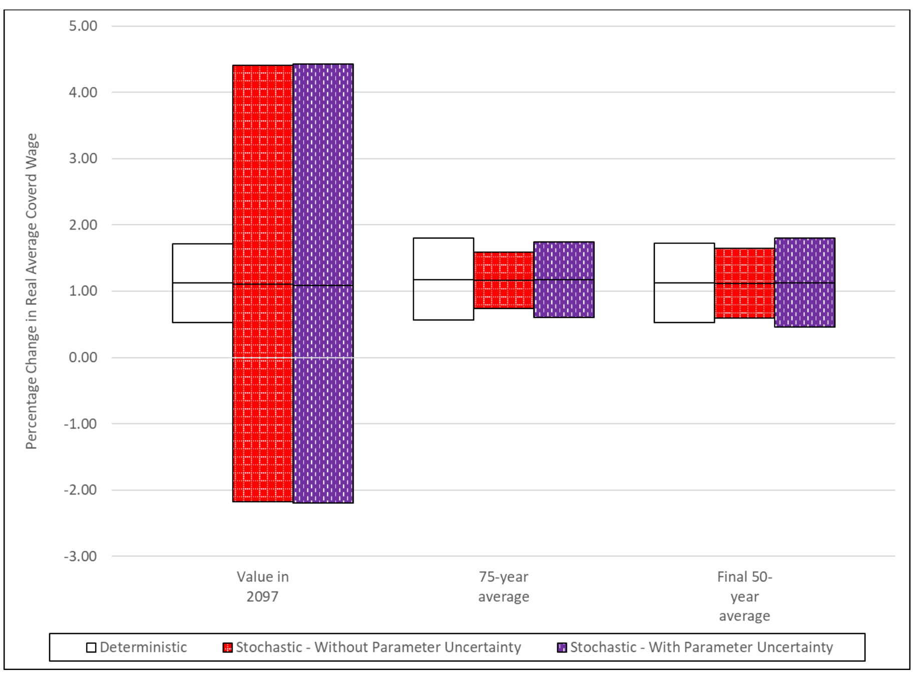


Figure IV.28a displays the real average covered wage throughout the entire historical and projection periods. Figure IV.28b displays the real average covered wage throughout the historical period and the cumulative average real average covered wage in the projection period.

Figure IV.28a – Real Average Covered Wage Annual Values, Calendar Years 1968-2097

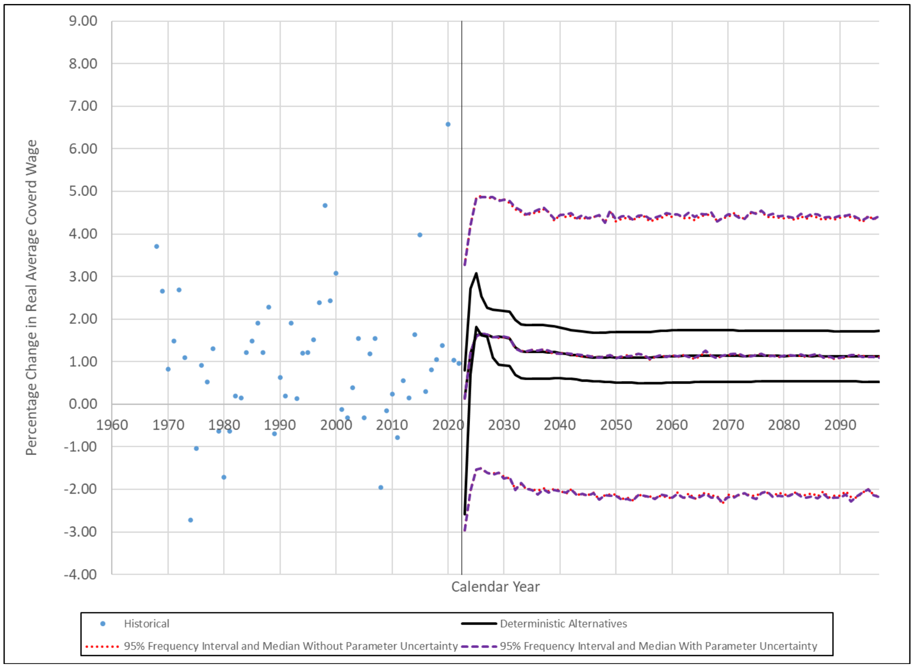
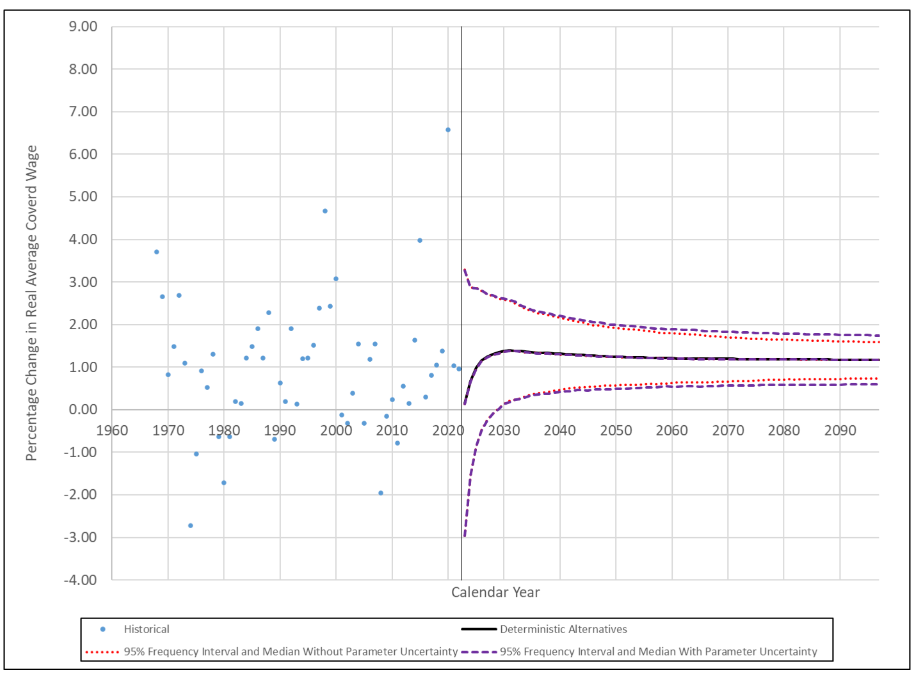


Figure IV.28b – Real Average Covered Wage Cumulative Average Values, Calendar Years 1968-2097



8. Disability Incidence Rate

Tables IV.14 and IV.15 show the results of the age-adjusted disability incidence rate equations for men and women, respectively. Figures IV.29 and IV.30 are a graphical display of tables IV.14 and IV.15, respectively. The disability incidence rates shown are per thousand disability-exposed individuals. Without parameter uncertainty, the 50-year average 95-percent frequency interval is larger than the alternative I and III range for men and about the same size for women but shifted up slightly due to the log-odds transformation of the incidence rates. With parameter uncertainty, the 95-percent frequency interval is notably wider than the alternative I and III range for men and slightly wider for women, and shifted upwards for both sexes.

Table IV.14 – Male Disability Incidence Rate (per thousand):
Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	4.98	4.99	4.98
	Alt I and III Interval	[3.96, 5.98]	[3.98, 5.98]	[3.96, 5.98]
Stochastic Model – Without Parameter Uncertainty	Median	4.96	5.11	5.13
	95% Frequency Interval	[2.84, 8.69]	[3.88, 6.70]	[3.61, 7.05]
Stochastic Model – With Parameter Uncertainty	Median	4.99	5.11	5.12
	95% Frequency Interval	[2.03, 11.52]	[2.52, 9.92]	[2.34, 10.40]

Table IV.15 – Female Disability Incidence Rate (per thousand):
Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	5.07	5.08	5.07
	Alt I and III Interval	[4.03, 6.08]	[4.05, 6.09]	[4.03, 6.08]
Stochastic Model – Without Parameter Uncertainty	Median	5.08	5.19	5.19
	95% Frequency Interval	[3.11, 8.18]	[4.33, 6.16]	[4.16, 6.40]
Stochastic Model – With Parameter Uncertainty	Median	5.08	5.20	5.20
	95% Frequency Interval	[2.92, 8.78]	[3.80, 7.00]	[3.67, 7.24]

Figure IV.29 – Male Disability Incidence Rate:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]

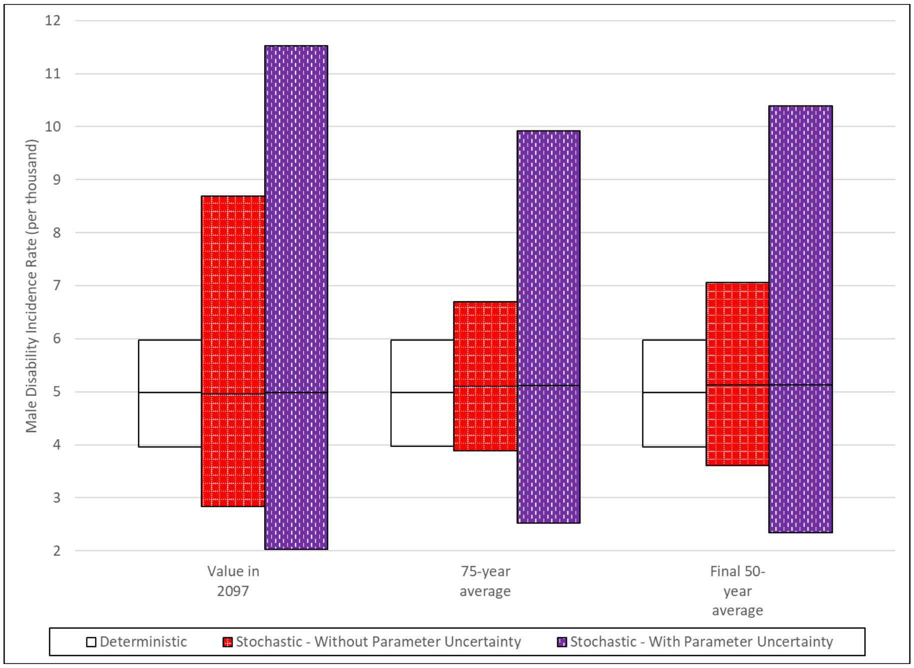
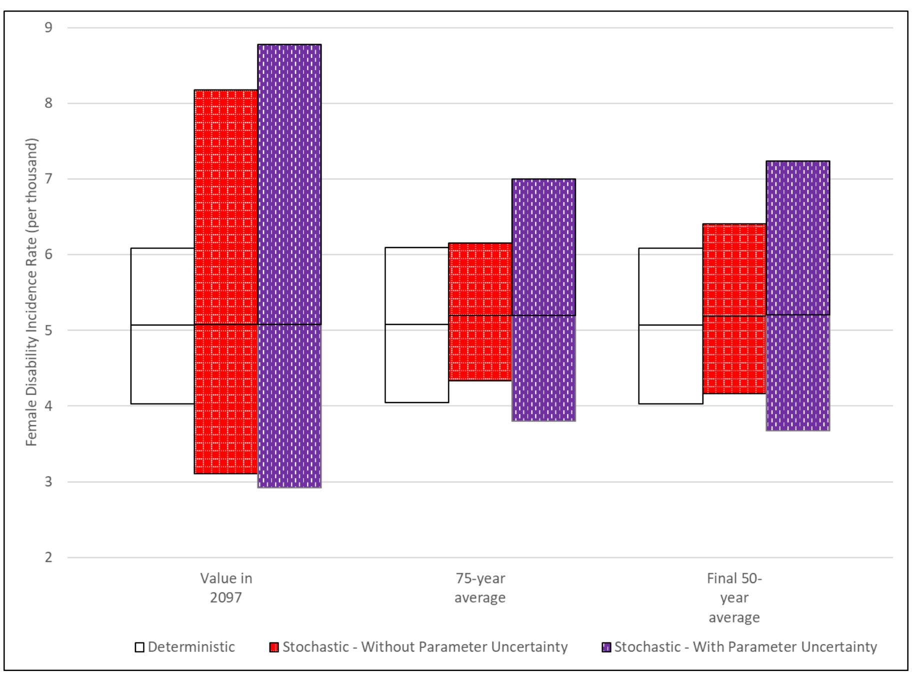


Figure IV.30 – Female Disability Incidence Rate:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]



Figures IV.31a and IV.32a display the disability incidence rate for men and women, respectively, throughout the entire historical and projection periods. Figures IV.31b and IV.32b display the disability incidence rate for men and women, respectively, throughout the historical period and the cumulative average disability incidence rate in the projection period.

Figure IV.31a – Male Disability Incidence Annual Rate, Calendar Years 1970-2097

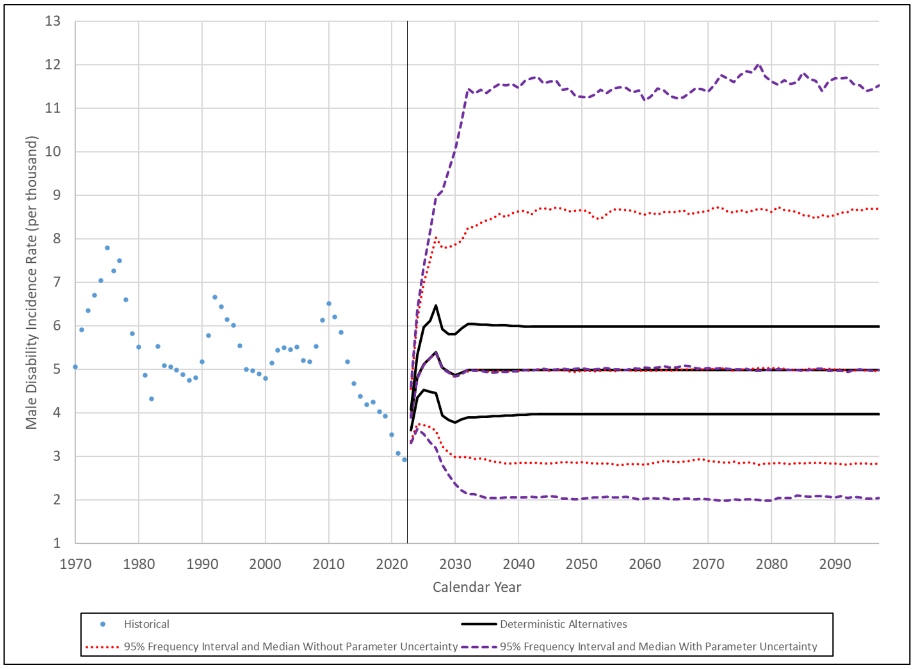


Figure IV.32a – Female Disability Incidence Annual Rate, Calendar Years 1970-2097

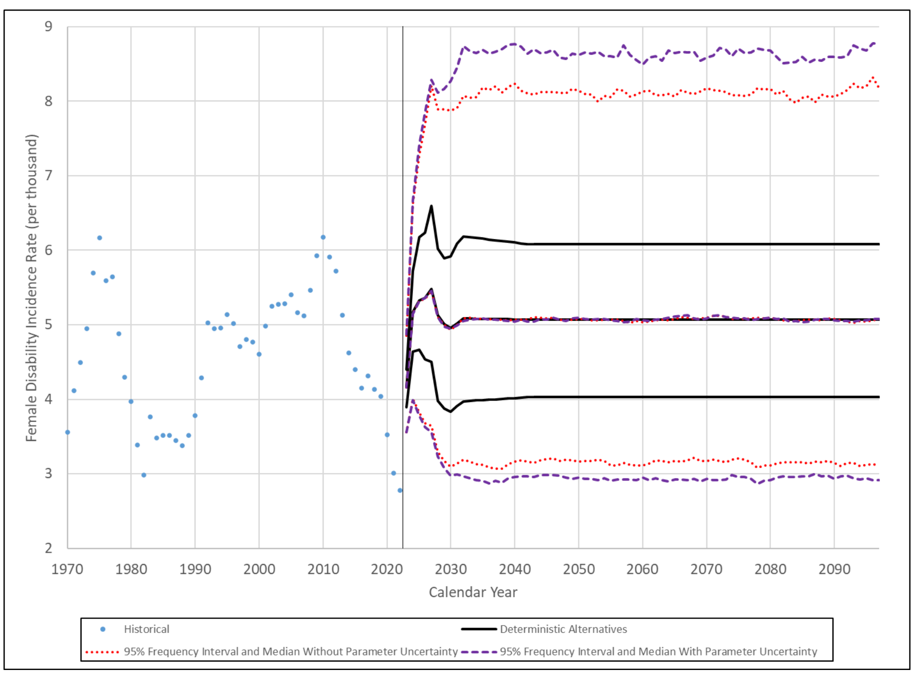


Figure IV.31b – Male Disability Incidence Cumulative Average Rate, Calendar Years 1970-2097

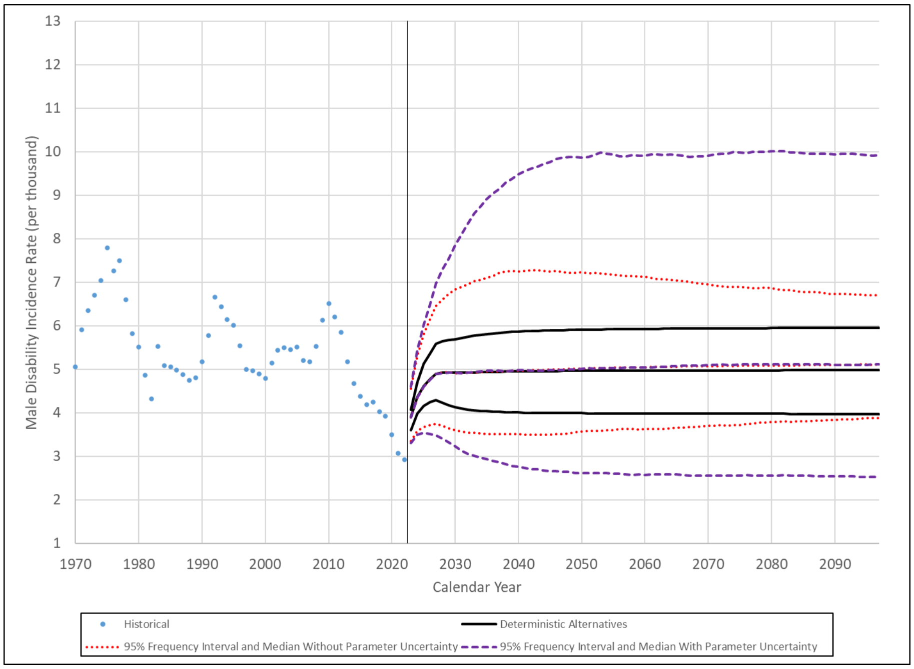
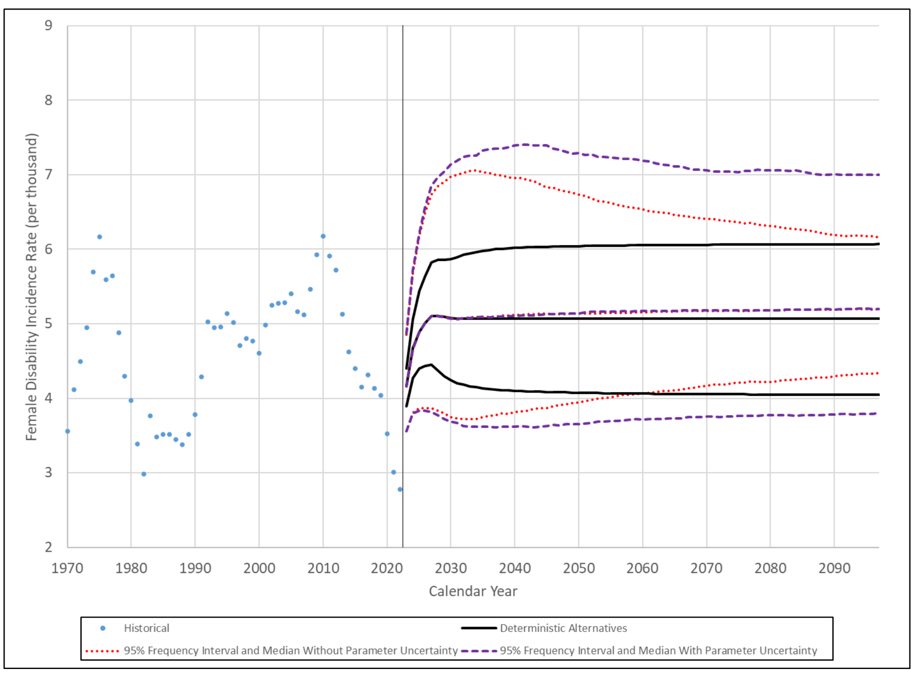


Figure IV.32b – Female Disability Incidence Cumulative Average Rate, Calendar Years 1970-2097



9. Disability Recovery Rate

Tables IV.16 and 17 show the results of the disability recovery rate equations for men and women, respectively. Figures IV.33 and IV.34 display tables IV.16 and IV.17 graphically, respectively. The disability recovery rates presented are per thousand disabled worker beneficiaries. Without parameter uncertainty, the 50-year average 95-percent frequency interval is slightly narrower for men and approximately the same for women when compared to each respective alternative I and III range. In addition, the median is slightly higher without parameter uncertainty due to the log-odds transformation of the disability recovery rates. With parameter uncertainty, the 95-percent frequency intervals are a bit wider than the alternative I and III ranges and still shifted upwards.

Table IV.16 – Male Disability Recovery Rate (per thousand): Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	9.62	9.86	9.64
	Alt I and III Interval	[7.61, 11.69]	[7.95, 11.77]	[7.62, 11.69]
Stochastic Model – Without Parameter Uncertainty	Median	9.76	10.28	10.01
	95% Frequency Interval	[5.21, 17.05]	[8.91, 11.71]	[8.42, 11.70]
Stochastic Model – With Parameter Uncertainty	Median	9.70	10.27	9.98
	95% Frequency Interval	[5.05, 17.76]	[7.99, 13.13]	[9.98, 13.13]

Table IV.17 – Female Disability Recovery Rate (per thousand): Selected Values and Intervals Under the Deterministic and Stochastic Models

		Value for 2097	75-year average	Final 50-year average
Deterministic Model	Alt II	8.73	8.92	8.73
	Alt I and III Interval	[6.92, 10.55]	[7.20, 10.63]	[6.92, 10.54]
Stochastic Model – Without Parameter Uncertainty	Median	8.76	9.32	9.07
	95% Frequency Interval	[4.07, 15.82]	[7.84, 10.91]	[7.39, 11.03]
Stochastic Model – With Parameter Uncertainty	Median	8.79	9.31	9.05
	95% Frequency Interval	[4.42, 16.87]	[6.89, 12.52]	[6.44, 12.63]

Figure IV.33 – Male Disability Recovery Rate:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]

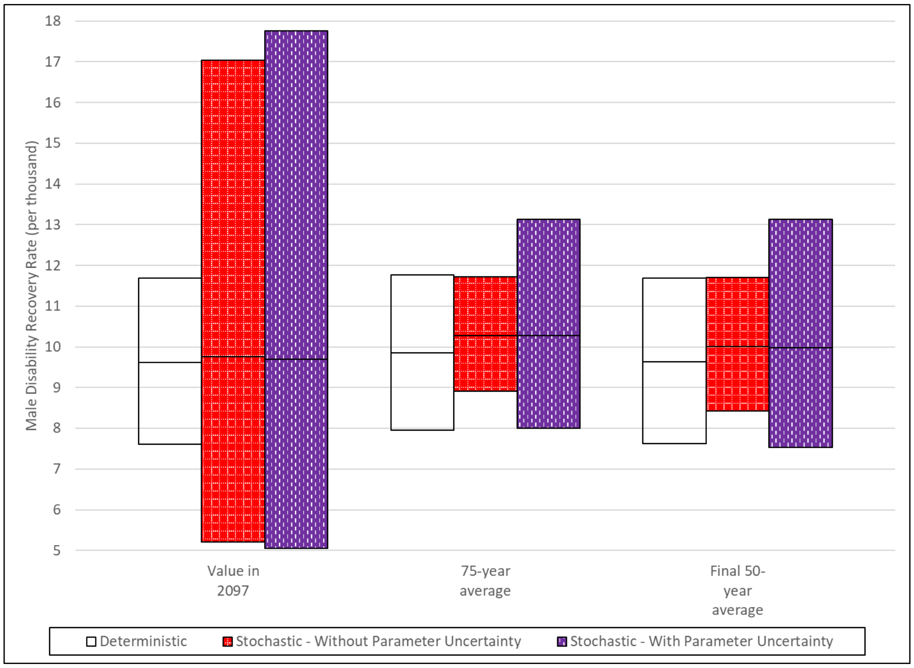
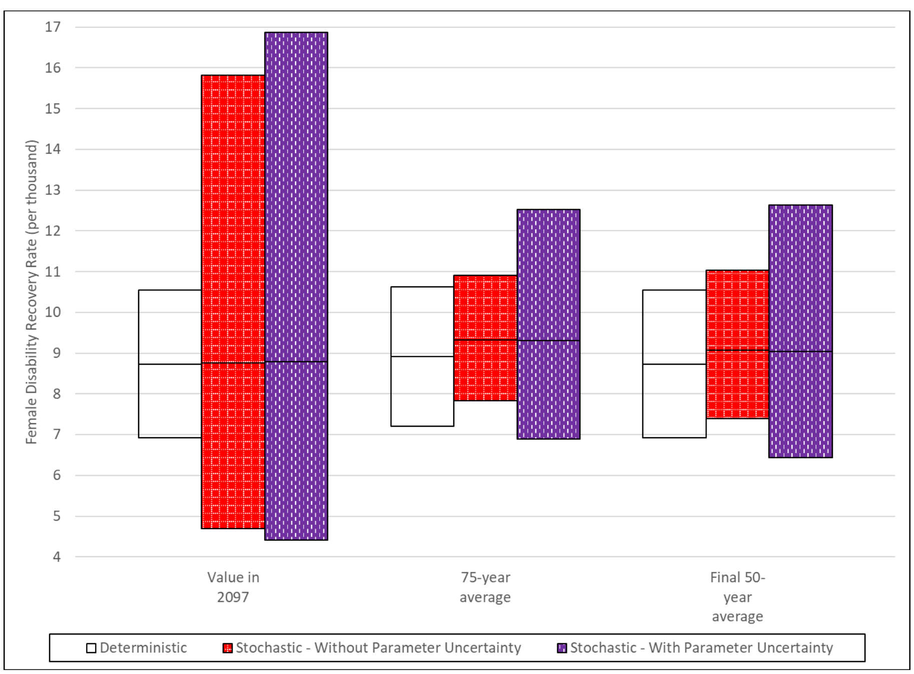


Figure IV.34 – Female Disability Recovery Rate:
Selected Values and Intervals Under the Deterministic and Stochastic Models
[Deterministic interval is defined by alternatives I and III;
stochastic intervals are the 95-percent frequency intervals]



Figures IV.35a and IV.36a display the disability recovery rate for men and women, respectively, throughout the entire historical and projection periods. Figures IV.35b and IV.36b display the disability recovery rate for men and women, respectively, throughout the historical period and the cumulative average disability recovery rate in the projection period.

Figure IV.35a – Male Disability Recovery Annual Rate, Calendar Years 1984-2097

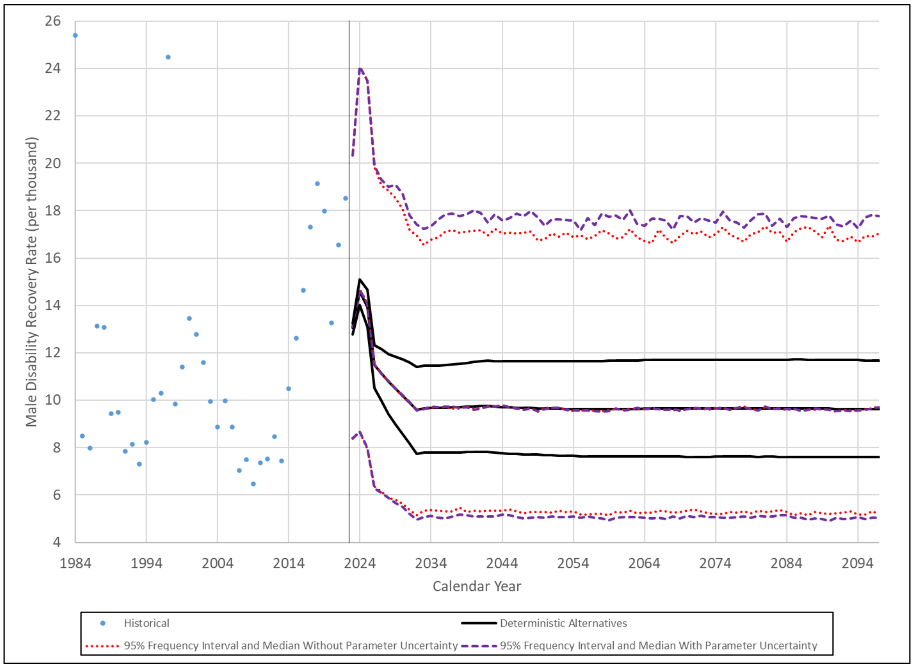


Figure IV.36a – Female Disability Recovery Annual Rate, Calendar Years 1984-2097

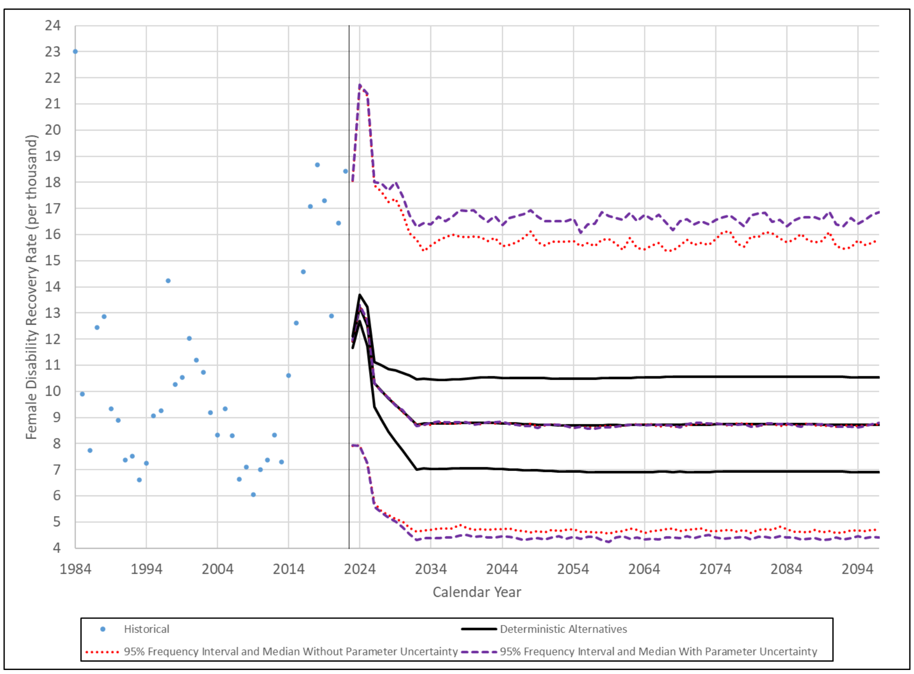


Figure IV.35b – Male Disability Recovery Cumulative Average Rate, Calendar Years 1984-2097

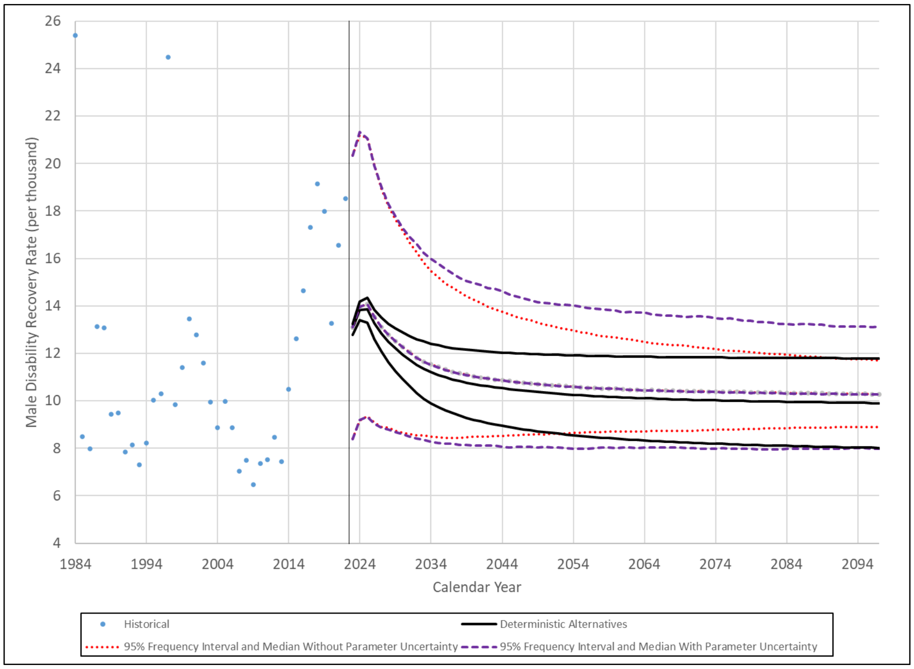
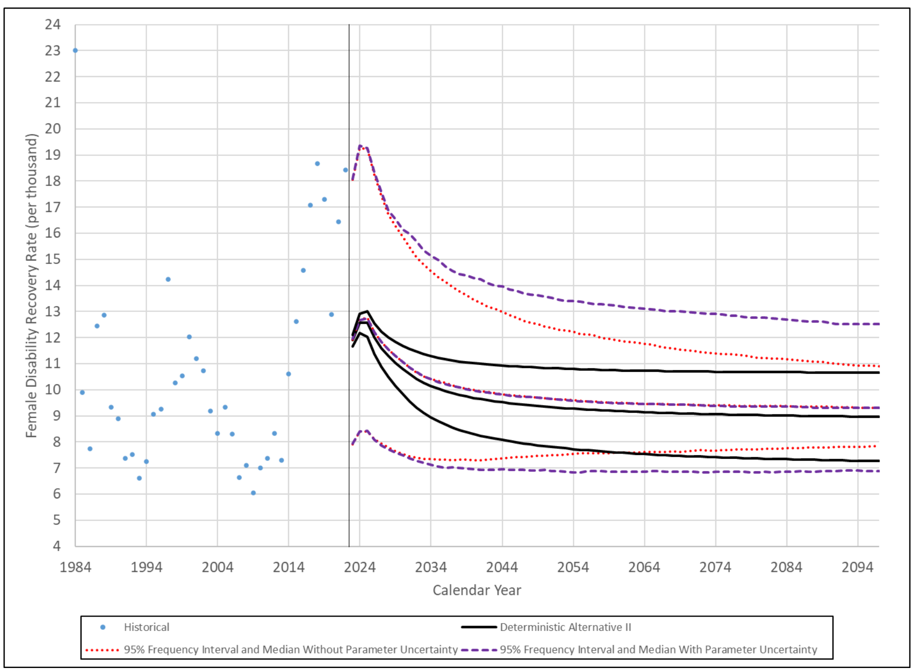


Figure IV.36b – Female Disability Recovery Cumulative Average Rate, Calendar Years 1984-2097



B. ACTUARIAL ESTIMATES

The tables in the following subsections contain estimates for the OASDI program as a whole. Each column in the tables contains estimates of a particular measure. For each measure, the rows show results from the 2023 Trustees Report deterministic model, the OSM without parameter uncertainty, and the OSM with parameter uncertainty. The Trustees Report intermediate assumption (alt II) value is shown followed by the low-cost alternative (alt I) and high-cost alternative (alt III) range. Then, the median is shown followed by the 95-percent frequency interval without parameter uncertainty. Finally, the median is shown followed by the 95-percent frequency interval with parameter uncertainty.

1. Annual Measures

Table IV.18 shows selected measures of the OASDI program for the 75th projection year, 2097. The median estimates shown from the OSM are slightly more pessimistic (from the perspective of the OASDI program) for results with and without parameter uncertainty than those from the Trustees Report intermediate alternative. Without parameter uncertainty, the 95-percent frequency interval is narrower than the low-cost alternative and high-cost alternative range for cost rate as a percentage of payroll and slightly wider than the low-cost alternative and high-cost alternative range for cost rate as a percentage of gross domestic product (GDP) and beneficiaries per 100 workers. With parameter uncertainty, the 95-percent frequency interval is slightly wider than the low-cost alternative and high-cost alternative range for cost rate as a percentage of payroll and wider than the low-cost alternative and high-cost alternative range for cost rate as a percentage of GDP and beneficiaries per 100 workers.

Table IV.18 – Estimates of the OASDI Program, Calendar Year 2097:
Selected Values and Intervals Under the Deterministic and Stochastic Models

		Cost Rate (% Payroll)	Cost Rate (% GDP)	Beneficiaries per 100 Workers
Deterministic Model	Alt II	17.75	5.98	47
	Alt I and III Interval	[12.34, 27.48]	[4.52, 8.54]	[36, 66]
Stochastic Model – Without Parameter Uncertainty	Median	18.32	6.16	50
	95% Frequency Interval	[13.37, 25.99]	[4.54, 8.62]	[37, 69]
Stochastic Model – With Parameter Uncertainty	Median	18.25	6.14	50
	95% Frequency Interval	[12.52, 27.93]	[4.25, 9.21]	[35, 74]

Figure IV.37a shows the estimated probability distribution of the annual trust fund ratio for the OASDI program from the OSM without parameter uncertainty, and figure IV.37b shows the estimated probability distribution of the annual trust fund ratio for the OASDI program from the OSM with parameter uncertainty. Note that the smooth lines that result do not represent the path of any particular simulation. Instead, for each given year, the curved lines represent the distribution of trust fund ratios based on all stochastic simulation results for that year. The two extreme curves in this figure comprise a 95-percent frequency interval.[29] Additionally, the median is shown along with the curves that bound the 80-percent frequency interval.

Figure IV.37a – Annual Trust Fund Ratios, Calendar Years 2023-97, Without Parameter Uncertainty

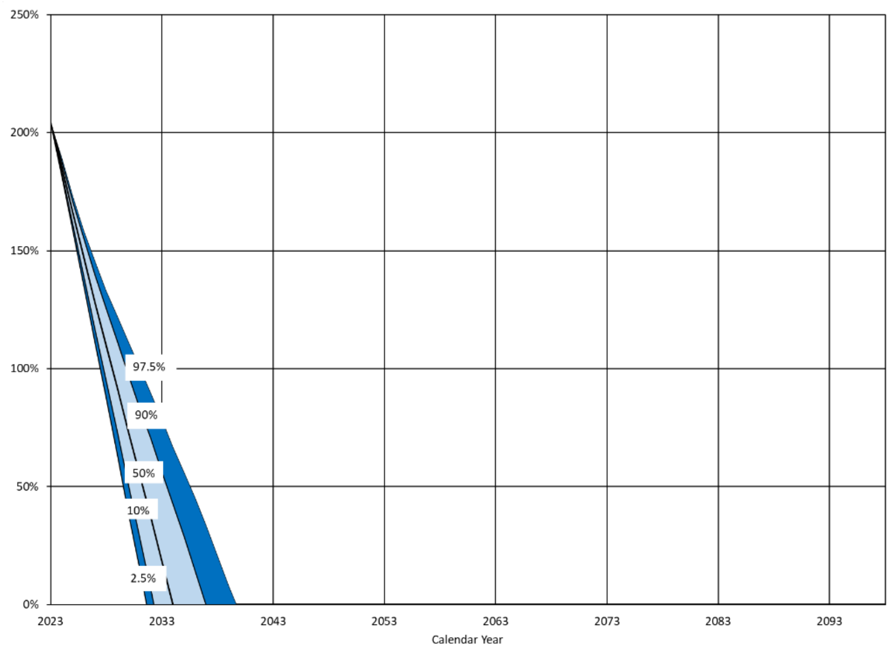
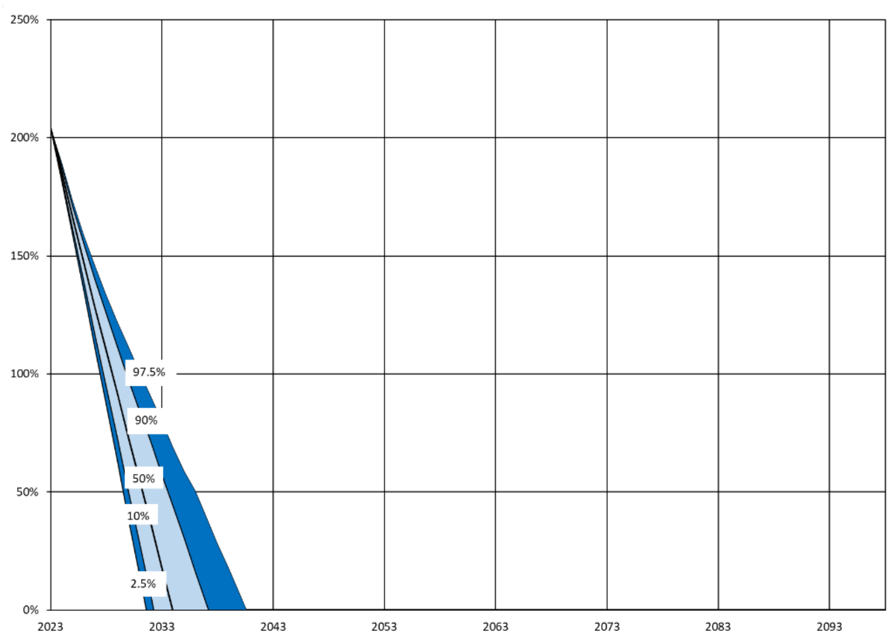


Figure IV.37b – Annual Trust Fund Ratios, Calendar Years 2023-97, With Parameter Uncertainty



[29] Note that bounds of the 95-percent frequency interval are the 2.5th and 97.5th percentiles.

An estimate of the depletion year of the combined OASI and DI Trust Fund reserves can be obtained from figures IV.37a and IV.37b by inspecting the *x*-intercept of a given curve. The *x*-intercept of the median curve indicates that according to the OSM (both without and with parameter uncertainty), there is about the same chance of trust fund reserve depletion prior to 2033 as there is after 2033. This depletion year is one year less than the reserve depletion year indicated by the 2023 Trustees Report intermediate alternative. Furthermore, the lower limit of the 95-percent frequency interval of the trust fund reserve depletion year indicated by the OSM with and without parameter uncertainty is also the same as the year in the 2023 Trustees Report high-cost alternative (2031). However, the upper limit as indicated by the OSM (2040 with parameter uncertainty and 2039 without parameter uncertainty) is less optimistic without parameter uncertainty than that of the 2023 Trustees Report low-cost alternative (2067).

Figures IV.38a and IV.38b show the estimated probability distribution of the annual cost rates for the OASDI program from the OSM without parameter uncertainty and with parameter uncertainty, respectively. The figures also include the income rates for the intermediate scenario. No distribution for the income rates is shown because there is relatively little variation in income rates across the 5,000 stochastic simulations. As before, the smooth curves which result do not represent the path of any particular simulation. Once again, the two extreme curves in this figure comprise a 95-percent frequency interval and the additional curves depict the median and the 80-percent frequency interval.

Figure IV.38a – Annual Cost and Income Rates, Calendar Years 2023-97, Without Parameter Uncertainty

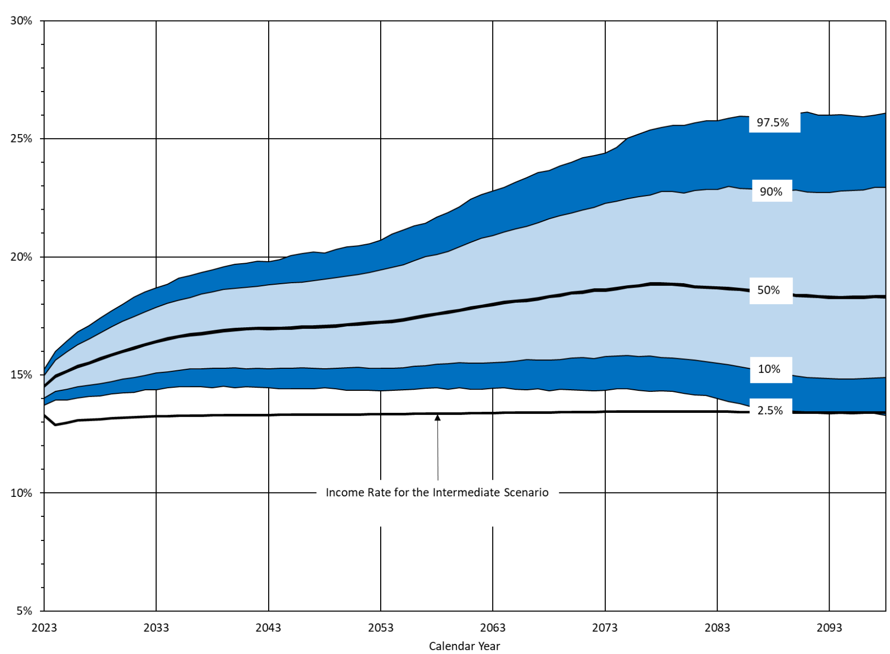
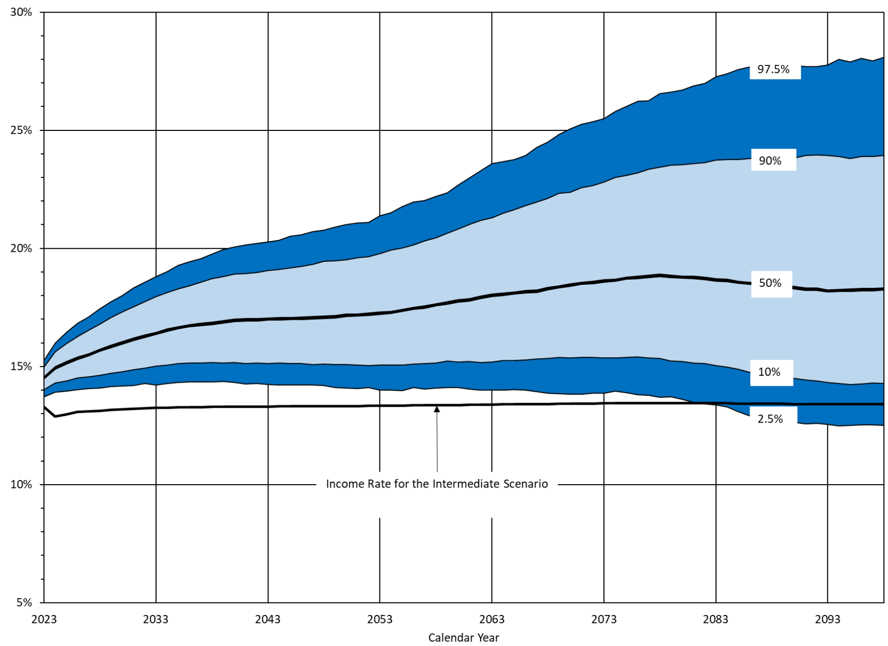


Figure IV.38b – Annual Cost and Income Rates, Calendar Years 2023-97, With Parameter Uncertainty



2. Summary Measures

Table IV.19 shows selected measures of the OASDI program for the 75-year projection period as a whole. The median estimates shown from the OSM are, in general, close to those from the 2023 Trustees Report intermediate alternative. Estimates with parameter uncertainty have wider ranges than estimates without parameter uncertainty.

Table IV.19 – Estimates of the OASDI Program:
Selected Values and Intervals Under the Deterministic and Stochastic Models

		Actuarial balance	Summarized cost rate	Summarized income rate	Open-group unfunded obligation (in trillions of dollars)	First projected year cost exceeds non-interest income and remains in excess through 2097	First year trust fund reserves become depleted
Deterministic Model	Alt II	-3.61	17.38	13.78	22.42	2023	2034
	Alt I and III Interval	[-8.37, -0.10]	[13.59, 22.52]	[13.49, 14.16]	[-0.43, 42.90]	[2023, [1]]	[2031, 2067]
Stochastic Model – Without Parameter Uncertainty	Median	-3.72	17.52	13.79	23.22	2023	2033
	95% Frequency Interval	[-6.39, -1.68]	[15.35, 20.31]	[13.55, 14.07]	[9.03, 51.37]	[2023, 2096]	[2031, 2039]
Stochastic Model – With Parameter Uncertainty	Median	-3.69	17.50	13.80	22.85	2023	2033
	95% Frequency Interval	[-7.17, -1.18]	[14.85, 21.09]	[13.49, 14.14]	[5.83, 60.85]	[2023, [1]]	[2031, 2040]

[1] Cost is projected to exceed non-interest income for a temporary period, before falling below non-interest income by the end of the projection period.

Figure IV.39 shows a frequency distribution of the long-range actuarial balances estimated from the OSM with and without parameter uncertainty. The width of each interval on the figure is 0.2 percent and the actuarial balance is expressed as a percentage of taxable payroll. It is interesting to note that none of the 5,000 simulations resulted in a positive long-range actuarial balance without parameter uncertainty, and only 5 of the 5,000 simulations resulted in a positive long-range actuarial balance with parameter uncertainty. In other words, according to the OSM, there is no measurable probability that the OASDI program has a positive actuarial balance for the period 2023-97 without parameter uncertainty and just a 0.1 percent probability with parameter uncertainty.

Figure IV.39 – Actuarial Balance Percentage Frequency Distribution of 5,000 Simulations

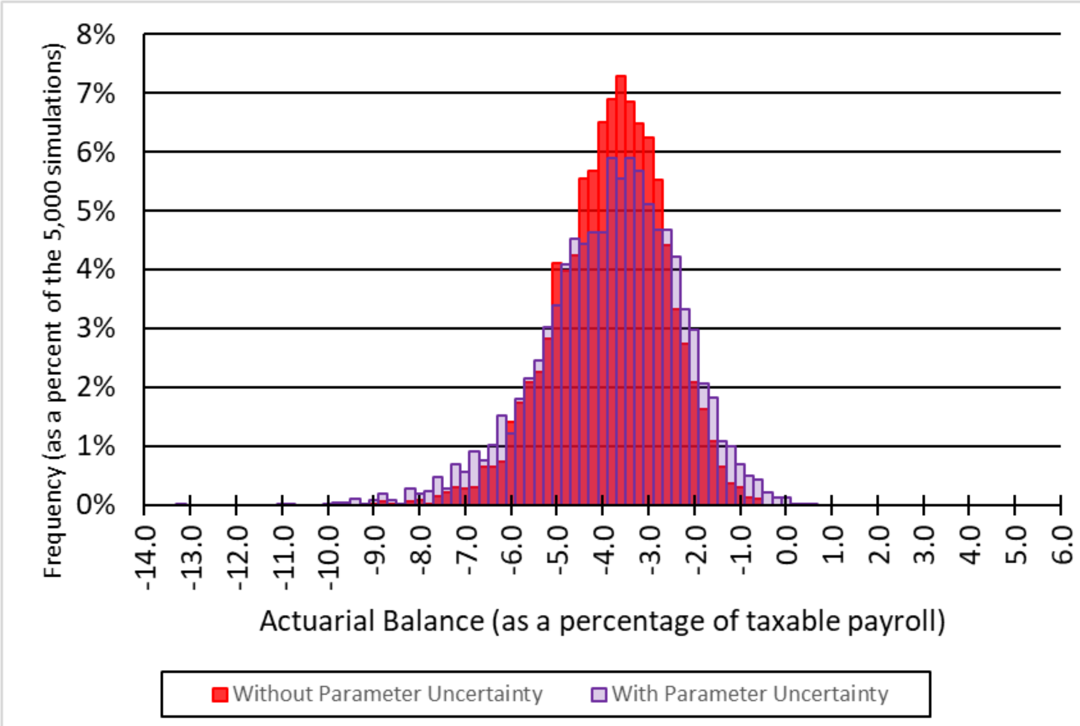
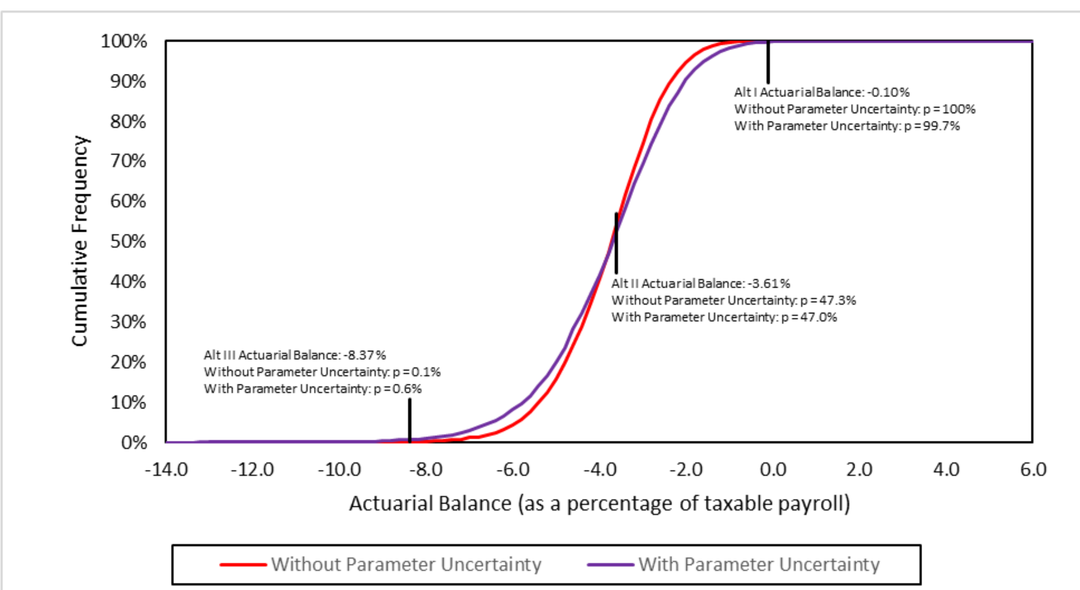


Figure IV.40 shows a cumulative frequency distribution of the long-range actuarial balances estimated from the OSM. The scale on the horizontal axis is the same as in figure IV.39. The 2023 Trustees Report high-cost, intermediate, and low-cost alternatives are each identified on the figure and labeled with the probability (according to the OSM) that the actuarial balance is less than or equal to this level.

Figure IV.40 – Actuarial Balance Cumulative Percentage Frequency Distribution of 5,000 Simulations



C. SENSITIVITY ANALYSIS

This section presents two types of sensitivity analysis for the OSM: stochastic and deterministic. A stochastic sensitivity analysis consists of performing a set of simulations in which a single equation (or group of related equations) is modeled stochastically while all other equations are modeled as for the 2023 Trustees Report intermediate alternative (alt II). A deterministic sensitivity analysis consists of a single simulation in which the mean of an equation (or group of related equations) is replaced with the 2023 Trustees Report low-cost (alt I) or high-cost (alt III) value, while the remaining equations are left with their means equal to the alt II values.

1. Stochastic Analysis

A stochastic sensitivity analysis is a process in which one, or some, of the demographic, economic, and programmatic variables are varied independently, while the other variables are set to their values under the 2023 Trustees Report intermediate alternative. This allows a measurement of how much of the variance of the final distributions can be attributed to each variable or set of variables.

Table IV.20 presents the median and average of the long-range actuarial balance, along with the standard deviation, for each of the stochastic sensitivity runs, both with parameter uncertainty and without. Each of these sensitivity runs is based on 5,000 simulations of the stochastic model. The first sensitivity run presented in table IV.20 allows only the total fertility rate to vary. The next sensitivity run varies only the 42 mortality improvement rates (21 age groups for boys/men, and 21 age groups for girls/women). The third run, listed as immigration, allows only the four immigration assumptions (level of LPR new arrival immigration, rate of legal emigration, rate of transfer from other-than-LPR to LPR status, and level of other-than-LPR immigration) to vary. Grouping the four economic assumptions (unemployment, inflation, real interest, and real average wage growth rates) and allowing only these variables to vary produces the economic sensitivity run. The last two sensitivity runs vary only the disability incidence rates and disability recovery rates, respectively.

Table IV.20 – Actuarial Balance Sensitivity to Varying Selected Assumptions

	Actuarial Balance[1]					
	Without Parameter Uncertainty			With Parameter Uncertainty		
	Median	Average	Standard Deviation	Median	Average	Standard Deviation
Effect of Changing Individual Assumption or Set of Assumptions:						
Fertility	-3.62	-3.66	0.70	-3.61	-3.66	0.82
Mortality Improvement	-3.71	-3.73	0.67	-3.72	-3.73	0.76
Immigration	-3.62	-3.62	0.07	-3.62	-3.62	0.12
Economic	-3.57	-3.58	0.61	-3.54	-3.55	0.84
Disability Incidence	-3.64	-3.65	0.19	-3.64	-3.69	0.42
Disability Recovery	-3.60	-3.60	0.01	-3.60	-3.60	0.02
Effect of Changing All Assumptions Together:	-3.72	-3.82	1.20	-3.69	-3.82	1.51

[1] The long-range actuarial balance under the 2023 Trustees Report intermediate alternative is -3.61 percent of taxable payroll.

2. Deterministic Analysis

A deterministic sensitivity analysis is a process in which the mean of one, or some, of the demographic, economic, and programmatic assumptions are set to equal the alternative I or the alternative III value, while the remaining variables are set to equal the alternative II values. In this case, the variance of all key input variables is set to zero and only one simulation is performed, as the zero variance ensures that the input variables for any simulation match the alternative I, II, or III values for those variables. This type of sensitivity analysis is similar to that done in the Trustees Report and can therefore be used to compare the underlying nature of the OSM and the deterministic model presented in the Trustees Report.

Table IV.21 compares the long-range actuarial balance results from the deterministic sensitivity run of the OSM to the results in the Trustees Report. The first row of each sensitivity grouping is from the sensitivity analysis in the Trustees Report, while the second row contains the sensitivity result from the OSM. Technically speaking, the first grouping ("All") is not really a sensitivity analysis, but instead shows the results of runs done using all of the assumptions based on alternatives I, II, and III. It is included as a point of reference. The next three results (fertility, mortality, and immigration) are the demographic deterministic sensitivity runs. These are followed by the three economic deterministic sensitivity runs (inflation, real interest, and real average wage). The final deterministic sensitivity run is programmatic (disability incidence). Note that unemployment and disability recovery sensitivity runs are not included because no comparable sensitivity analysis is performed for the Trustees Report.

Table IV.21 – Actuarial Balance Comparison of Deterministic and Stochastic Models

	Actuarial Balance		
Sensitivity:	Deterministic Low Cost	Deterministic Intermediate[1]	Deterministic High Cost
All			
Trustees Report	-0.10	-3.61	-8.37
OSM	-0.13	-3.61	-8.49
Fertility			
Trustees Report	-3.14	-3.61	-4.32
OSM	-3.13	-3.61	-4.34
Mortality			
Trustees Report	-2.91	-3.61	-4.39
OSM	-2.71	-3.61	-4.67
Immigration			
Trustees Report	-3.21	-3.61	-4.02
OSM	-3.22	-3.61	-4.02
Inflation			
Trustees Report	-3.47	-3.61	-3.74
OSM	-3.48	-3.61	-3.78
Real Interest			
Trustees Report	-3.43	-3.61	-3.79
OSM	-3.42	-3.61	-3.78
Real Average Wage			
Trustees Report	-2.44	-3.61	-4.81
OSM	-2.32	-3.61	-4.90
Disability Incidence			
Trustees Report	-3.25	-3.61	-3.98
OSM	-3.28	-3.61	-3.91

[1] This column does not show a sensitivity analysis but is shown for comparison purposes.

Table IV.21 shows that the actuarial balances produced by the OSM are generally similar to those from the deterministic model.

V. APPENDICES

A. DOCUMENTATION OF COMPUTER PROGRAM

This chapter describes the details of the computer program used to run the OSM.

1. Organization

The OSM contains nine modules. They are executed sequentially in the following order: Assumptions, Population, Economics, Insured, DI (Disability Insurance Beneficiaries), RSB (Old-Age and Survivors Insurance Beneficiaries), Awards, Cost, and Summary Results. In a sequential model, the output from an earlier module may become input to a later module. The flow of data among the OSM modules is summarized in table V.1. The first column lists the nine modules in the order in which they are executed. For each module, the second column lists the modules from which it receives input, while the third column lists the modules to which it provides input. For example, the Population Module receives input from only one module (i.e., Assumptions) and provides output (that then becomes input) to six modules (i.e., Economics, Insured, DI, RSB, Awards, and Cost). The Assumptions Module does not receive data from any of the other modules, and the Summary Results Module does not send data to any of the other modules. The only instance in which a module sends data to an earlier module is that the DI Module sends data to the Economics Module. This is possible because the DI data sent there are from the prior (not current) year within the simulation.

The computer program is organized to go through three main phases: initialization, simulation, and wrap-up. In the initialization phase, the program prepares input and output files and variables needed by each module. In the simulation phase, the program solves the first eight modules using two nested loops. The first (outermost) is the run-number loop. It loops once for each simulation. The second is the year loop. It loops from the first year of the simulation (the current Trustees Report year) through the last year of the simulation. In the wrap-up phase, the final module, Summary Results, sorts and prints the final output results.

Table V.1 – Module Dependencies

Module	Input Modules	Output Modules
Assumptions	N/A	Population Economics DI Cost Summary Results
Population	Assumptions	Economics Insured DI RSB Awards Cost Summary Results
Economics	Assumptions Population DI	Insured Awards Cost
Insured	Population Economics	DI RSB Awards
DI	Assumptions Population Insured	Economics RSB Cost
RSB	Population Insured DI	Awards Cost
Awards	Population Economics Insured RSB	Cost
Cost	Assumptions Population Economics DI RSB Awards	Summary Results
Summary Results	Assumptions Population Cost	N/A

2. Modules

All of the modules, with the exception of the Assumptions and Summary Results Modules, are adapted from the deterministic computer model used to prepare the 2023 Trustees Report. The modules are written so that the set of non-stochastic inputs required to begin the projections is identical to the input assumptions used when running the deterministic model under the 2023 Trustees Report intermediate alternative. Moreover, the mean value for each stochastic variable is set equal to the value assumed for the variable under the 2023 Trustees Report intermediate alternative.

a. Assumptions

The Assumptions Module contains 55 equations, one for each stochastic variable. The 55 stochastic variables are the total fertility rate, changes in mortality rates (21 male age groups and 21 female age groups), new arrival LPR and other-than-LPR immigration levels, rates of adjustments of status (from other-than-LPR to LPR), rates of legal emigration (from the population of citizens and LPRs), changes in the Consumer Price Index, changes in average real wages, unemployment rates, trust fund real interest rates, disability incidence rates (male and female), and disability recovery rates (male and female). These equations are described in detail in chapter III.

The equations are used to set the annual values for the stochastic variables. In any particular year, the value for a stochastic variable is determined, in part, by the equation's error term. If the error term for an equation is not dependent on the error terms of other equations, then a random number is drawn for each year from a normal distribution with mean zero and standard deviation equal to the estimated standard error for the equation. If the error term for an equation is dependent on the error terms of other equations, then a Cholesky decomposition is used to assign the appropriate level of covariance. See appendix C for more details on this process.

The final step of the Assumptions Module is to use the error terms to calculate the results of each equation. Chapter III provides more details about these calculations for each equation.

For the mortality equations, there is an additional step to decompose the annual rates of decrease in the central death rates by age group into single years of age.

b. Population

The Population Module projects the Social Security area population by sex, single year of age, and marital status. The Population Module also projects the other-than-LPR population. The components of change—fertility, mortality, and immigration—are applied each year throughout the projection period based on levels and rates generated in the Assumptions Module. The population is grouped by marital status using the relative proportions for each age-sex group.

The population is projected by starting with the beginning of the year population, adding births and immigration, and subtracting deaths and emigration. The total fertility rate is distributed among women of childbearing age using the relative proportions of age-specific birth rates for each year. The age-specific birth rates are then applied to the midyear population to calculate the number of births. For the mortality projection, central death rates are computed by applying the rates of decrease in the single year of age central death rates to the previous year's central death rates. Death probabilities are derived from the central death rates by assuming a uniform distribution of deaths for each age (except age 0). The death probabilities are then applied to the beginning of the year population to calculate the number of deaths for each single year of age and sex group. For each type of immigration, the annual levels are distributed among the age-sex groups by using the relative proportions. The resulting population is then distributed by marital status using the relative marital proportions for each age-sex group.

c. Economics

The Economics Module receives data from the Assumptions, Population, and DI Modules. The Assumptions Module passes civilian unemployment and inflation rates, along with the growth rate in the real average covered wage. The Population Module passes the age-sex levels of the Social Security area population and their life expectancies. The DI Module passes one-year-lagged age-sex levels of disabled-worker beneficiaries in current-payment status.

For employment-related variables, the module projects various measures for the total U.S. economy and then converts them to OASDI covered concepts. For the earnings variables, the module initially projects OASDI covered wages then converts them to a U.S. economy-wide concept. The module estimates levels for most key variables by projecting deviations from values produced for the intermediate assumptions.

Labor Force Participation

Future civilian labor force participation rates by age and sex are influenced by projected disability prevalence rates, current and lagged unemployment rates, and life expectancies. For a given year, the civilian labor force is summed from the products of the civilian labor force participation rates and civilian noninstitutional populations by age and sex. For each age-sex group, the civilian noninstitutional population is the product of the Social Security area population and the ratio of the noninstitutional population to the Social Security area population.

Employment

The civilian unemployment rates by age and sex are projected by distributing the stochastically projected aggregate rate to its age-sex components. Projected economy-wide employment by age and sex is derived from the corresponding components of the civilian labor force and unemployment rates. The concept of economy-wide employment represents an "average" level of employment for a particular year. The projected economy-wide employment by age and sex is then used to estimate the number of workers with employment at any time during the year, a concept closer to OASDI covered employment.

Covered Employment

Total OASDI covered employment by age and sex is projected by removing non-covered workers, such as certain categories of government employees, foreign workers in certain visa categories, and undocumented immigrants, from the total at-any-time employment. Total OASDI covered employment is then broken down into those with wages and those with self-employed income only.

Covered Wages

Total OASDI covered wages are the product of the number of covered wage workers and their average nominal covered wage. The average nominal covered wage is determined using the annual inflation rate and the real average OASDI covered wage from the Assumptions Module. Total U.S. economy-wide wages are projected as a ratio to OASDI covered wages, adjusted for relative differences in other-than-LPR immigration. Total compensation for wage workers, total and covered self-employed income, taxable wages, and taxable self-employed net income are all derived from assumed relationships in the intermediate alternative. Multi-employer refund wages are projected as a ratio to OASDI covered wages, adjusted for relative differences in the unemployment rate.

Taxable Payroll, Average Wage Index, and Cost of Living Adjustment (COLA)

The OASDI taxable payroll is the sum of taxable wages and self-employed income, less one-half of multi-employer refund wages. The average wage index is determined using the annual growth rate in the economy-wide average wage, defined as the ratio of total U.S. economy-wide wages to total at-any-time wage employment. The COLA is determined by the inflation rate.

d. Insured (Fully Insured and Disability Insured)

Fully insured status is required to receive worker benefits and is determined by a worker's accumulation of quarters of coverage (QCs). Prior to 1978, one QC was credited for each calendar quarter in which at least $50 was earned. Quarterly reporting was replaced by annual reporting in 1978. The minimum annual required amount, starting with $250 for each QC in 1978, is adjusted each year according to the average wage index. This value for 2023 is $1,640. Thus, if a worker earns at least $6,560 in covered employment any time during 2023, then the worker receives credit for four quarters of coverage.

Fully insured status is determined by the number of earned QCs and the worker's age. To be fully insured, a worker must have a total number of QCs greater than or equal to the number of years elapsed after attaining age 21 (with a minimum of six QCs required). Once reaching 40 QCs, the worker remains permanently fully insured. Disability insured status is acquired by any fully insured worker over age 30 who has accumulated 20 QCs during the 40-quarter period ending with the quarter in which the disability began. A fully insured worker aged 24-30 needs to accumulate at least one-half of the quarters elapsed after attaining age 21. A fully insured worker under age 24 needs to have accumulated six QCs during the 12-quarter period immediately before becoming disabled.

In the intermediate alternative, projections of the fully insured population, as a percentage of the Social Security area population, are made by age and sex for each birth cohort beginning with 1900. Percentages for everyone except the other-than-LPR immigrant population are based on 30,000 simulated work histories for each sex and birth cohort. The simulated work histories are constructed to reproduce fairly closely the historical insured percentages from 1990 to date, using the historical portions of the following data:

- Median earnings, by age and sex,
- Covered workers and Social Security area population, by age and sex, and
- Net LPR immigrants and other-than-LPR immigrants, by age and sex.

The projected portions of the above data are then used to extend the simulation of work histories throughout the projection period. Projected fully insured percentages for each sex and birth cohort are then determined by identifying all simulated work histories that meet the QC requirement for fully insured status as a percentage of the 30,000 simulated cases, which represent everyone except the other-than-LPR immigrant population. For the other-than-LPR immigrant population, the model generates substantially lower percentages of individuals attaining fully insured status. A similar process is applied to produce the disability insured percentages.

In the OSM, the Insured Module projects the percentages of the population that will be fully insured and disability insured for each birth-sex cohort. For everyone except the other-than-LPR immigrant population, projections of fully insured percentages are based on the baseline projection in the intermediate alternative and an adjustment that accounts for the difference between the 10-year moving averages of the covered worker rates[30] from the Economics Module and the intermediate alternative. A small portion of the other-than-LPR immigrant population is then added to calculate the fully insured rate of the Social Security area population. Projections of the disability insured percentages are modeled in a similar manner. Finally, these percentages are multiplied by the Social Security area population from the Population Module to produce the numbers of insured individuals.

e. Disability Insurance Beneficiaries (DI)

The DI Module begins with projections of the disabled-worker beneficiaries in current-payment status. The projections are based on the age-sex specific disability insured population passed from the Insured Module, the age-sex specific mortality rates passed from the Population Module, and the age-adjusted male and female incidence and recovery rates passed from the Assumptions Module. Additionally, the DI Module estimates[31] the number of those currently entitled (as of the beginning of the projection period) to a disabled-worker benefit as a starting value.

Disabled Workers

The number of disabled-worker beneficiaries at the end of a year is calculated by adding those newly entitled to a disabled-worker benefit during the year to those currently entitled at the beginning of the year and subtracting those who recover, die, or convert to a retirement benefit upon reaching normal retirement age during the year. New entitlements are calculated by multiplying the incidence rate by the exposed population (disability insured less those currently entitled). For each sex, the future age-specific incidence rates are assumed to grow at the same rate as the growth in the age-adjusted incidence rate.[32]

Deaths and recoveries are calculated by applying the death and recovery rates to the number of people who are currently entitled at the beginning of the year and to the number of people who are newly entitled during the year. Death rates by age, sex, and duration since entitlement are projected to improve at the same rate as the general population aged 15 through 64. For each sex, the future

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[30] Covered worker rates are defined as the number of covered workers, expressed as a percentage of the Social Security area population.

[31] The current number of entitled disabled-worker beneficiaries is not completely known because of the time lag between entitlement to and receipt of benefits.

[32] Growth is determined relative to a 2005-14 base period for ages 15-60. Rates for ages 60 and over include adjustments for the increase in normal retirement age.

age-specific recovery rates are assumed to grow at the same rate as growth in the age-adjusted recovery rate.[33] The number of disabled-worker beneficiaries in current-payment status is then estimated by reducing the number of those currently entitled by the number of those for whom payment has not yet begun.

Dependents of Disabled Workers

The projected number of auxiliary beneficiaries of disabled workers depends on the projections of disabled workers and the Social Security area population. Minor child beneficiaries of disabled workers are projected as the product of the child population and factors which represent the probabilities that a worker is under normal retirement age, is disability insured, and is disabled, and a statistical residual factor. Student and disabled-adult-child beneficiaries are calculated similarly. Married aged-spouse beneficiaries of disabled workers are projected as a percentage of disabled-worker beneficiaries. This percentage is set as in the intermediate alternative and a factor that adjusts for differences between the OSM and the intermediate alternative projected distributions of the age 62 or older married population. Young-spouse and divorced aged-spouse beneficiaries are calculated similarly, but with their respective populations.

f. Old-Age and Survivors Insurance Beneficiaries (RSB)

The RSB module receives variables passed from the Population, Insured, and DI Modules. The Population Module passes the Social Security area population by age, sex, and marital status. The Insured Module passes the number of fully insured persons, by age, sex, and marital status, and also passes the LPR fully insured rates by age and sex. The DI Module passes disability prevalence rates[34] and the numbers of disabled-worker and converted disabled-worker beneficiaries, by age and sex. Using these data, the RSB Module estimates the number of retired-worker beneficiaries, along with five categories of auxiliary and survivor beneficiaries who are eligible to receive benefits based on the earnings of a retired or deceased worker (also referred to as the primary account holder). These categories are aged-widow(er), aged-spouse, disabled-widow(er), children (minor, student, and disabled adult), and young-spouse beneficiaries.

Retired Workers

To calculate the number of retired-worker beneficiaries, the population aged 62 or older is multiplied by the probability that:

- The worker is fully insured,
- The worker is not receiving disability benefits, and
- The worker is not an insured aged widow(er).[35]

[33] Growth is determined relative to a 2011-15 base period.

[34] A disability prevalence rate is the ratio of the number of disabled workers to the number of disability insured workers.

[35] In this case, the worker is fully insured and, therefore, eligible to receive his/her own retired-worker benefit. Instead, he/she has decided not to apply for a retired-worker benefit and is receiving only an aged-widow(er) benefit.

Retirement prevalence rates[36] used in the intermediate alternative are then applied to calculate the number of retired-worker beneficiaries. Due to the elimination of the file and suspend and deemed filing claiming strategies by the Bipartisan Budget Act of 2015, additional factors are included in the calculation of the prevalence rates. These factors increase the projected number of married and divorced retired workers to account for those who will no longer delay receiving a benefit as a part of the eliminated claiming strategies.

Aged Widow(er)s

Aged widow(er)s are divided into two subcategories: insured and uninsured. The number of insured aged-widow(er) beneficiaries is projected as the product of the widowed and divorced population aged 60 or older, and the probability that:

- The primary account holder is deceased,
- The primary account holder was fully insured at death, and
- The aged-widow(er) is fully insured but, if at least age 62, did not apply for a retired-worker benefit based on his/her own earnings (assuming that his/her own retired-worker benefit is less than his/her widow(er) benefit).

The number of uninsured aged-widow(er) beneficiaries is projected as the product of the widowed and divorced population aged 60 or older, and the probability that:

- The primary account holder is deceased,
- The primary account holder was fully insured at death,
- The aged widow(er) is not fully insured,
- The aged widow(er) is not receiving a young-spouse benefit for the care of a child, and
- The aged-widow(er)'s benefits are not withheld because of receipt of a significant government pension based on earnings in noncovered employment.

For both the insured and uninsured categories, an additional probability is applied which accounts for other conditions not previously mentioned. For example, in the case of an aged widow(er), the additional factor includes the probability that the widow(er) did not remarry before age 60. In the case of a divorced widow(er), the factor includes the probability that the marriage to the primary account holder lasted at least 10 years.

Aged Spouses

The number of aged spouses of retired workers is projected as the product of the married and divorced population aged 62 or older, and the probability that:

- The primary account holder is alive and fully insured,

[36] A retirement prevalence rate is the ratio of the number of retired workers to the number of fully insured workers (not receiving disability or widow(er)'s benefits).

- The primary account holder is receiving a retirement benefit (not required for divorced spouses),
- The aged spouse is not receiving a young-spouse benefit for the care of a child,[37]
- The aged spouse is not insured,
- The aged spouse's benefits are not withheld because of receipt of a significant government pension based on earnings in noncovered employment, and
- The couple is engaging in a filing strategy in which the primary account holder files for their benefit and the fully insured spouse files only for their aged spouse benefit.

In addition to the stated conditions, an adjustment is made for other requirements. One such requirement is that the aged spouse has been married to the primary account holder for at least one year. In the case of a divorced aged spouse, the requirement is that their marriage had lasted at least 10 years. As is the case with many of the listed requirements, there are exceptions to this requirement.

Disabled Widow(er)s

To calculate the number of disabled-widow(er) beneficiaries, the widowed and divorced population ages 50 through 69 is multiplied by the probability that:

- The primary account holder is deceased,
- The primary account holder was fully insured at time of death,
- The surviving spouse is disabled, and
- The disabled widow(er) is not receiving another type of benefit.

Finally, an additional factor is applied to account for other eligibility requirements. For example, there is a 7-year deadline for surviving spouses to qualify for benefits on the basis of disability.

Children

The RSB Module calculates the number of child beneficiaries for three different child categories: minor, student, and disabled adult. Child beneficiaries are estimated separately for retired and deceased primary account holders. The population of potential beneficiaries for minor children includes children under age 18, while student status includes children of age 18 (and occasionally also age 19). Disabled adult status includes all disabled persons over age 17 who were disabled prior to age 22.

To calculate the number of minor children of male retired workers, the population of children under age 18 with a parent aged 62 and over whose other parent is not deceased is multiplied by the ratio of the number of retired workers aged 62 to 71 to the number of members of the population aged 62 to 71. We maintain the total number of minor children of female retired workers at the

[37] The upper age limit to be eligible for a young-spouse benefit is 69, as long as there is a dependent child under 16 or disabled. Since the youngest age to receive an aged-spouse benefit is 62, there is a chance that some spouses between the ages of 62 and 69 are still receiving young-spouse benefits.

five-year historical average percentage of minor children of female retired workers to the total number of minor children of male and female retired workers. We then use the proportion of beneficiaries at each age in the last historical year to distribute the minor children of female retired workers among the ages 0 to 17.

For minor children of deceased workers, the population of children under age 18 is multiplied by the probability that:

- The parent is deceased,
- The parent was fully or currently insured at time of death, and
- The child is not receiving a benefit based on his/her other parent's earnings.

Student and disabled adult children of retired and deceased workers are calculated similarly using their respective age-specific populations. The calculations for student and disabled children of retired workers have the additional condition that the parent is receiving a benefit.

For each child category, an adjustment is made for other conditions, such as the marital status of the child (more common in the case of a disabled adult child) or the dependency status of the child. For example, if a child marries, he/she is no longer entitled to a benefit. Also, if it is determined that the child is not dependent upon the parent (or was not at the time of the parent's death) then he/she is not entitled to receive benefits.

Young Spouses

Young-spouse beneficiaries are broken into two categories, young spouses of retired workers and young spouses of deceased workers (also referred to as mother-survivor and father-survivor beneficiaries). To estimate the number of young spouses of retired workers, the married population under age 70 is multiplied by the probability that:

- The primary account holder is age 62 or older,
- The young spouse has an entitled child (under age 16 or a disabled adult) in their care, and
- The young spouse is not already receiving benefits based on another child in their care.

To estimate the number of young spouses of deceased workers, the population of widowed and divorced spouses under age 70 is multiplied by the probability that:

- The primary account holder is deceased,
- The young spouse has an entitled child (under age 16 or a disabled adult) in their care,
- The young spouse is not already receiving benefits based on another child in their care, and
- The young spouse has not remarried.

As with all categories, an additional factor is applied to account for other eligibility requirements, such as ensuring that the young spouse is not entitled to a widow(er) benefit and is not receiving a retired-worker benefit based on his/her own earnings.[38]

g. Awards

The Awards Module uses a stratified sample of newly entitled worker beneficiaries with their earnings histories. A one-percent sample is used for OASI beneficiaries and a five-percent sample is used for DI beneficiaries. In addition, the Awards Module receives the number of covered workers (historical and projected) from the Economics Module, the population (historical and projected) from the Population Module, and fully insured rates (historical and projected) from the Insured Model. The Awards Module also utilizes other historical and projected data from the Economics Module, such as average wage and average taxable earnings, to produce projected taxable earnings used in the calculation of future awards. Additionally, the Awards Module utilizes the distribution of the number of in-current-pay retired beneficiaries by age at retirement from the RSB Module to project the shifting of worker retirement to later ages relative to the initial entitlement sample. The Awards Module ultimately produces projected levels of benefits, in terms of average indexed monthly earnings (AIME), for those beneficiaries newly entitled by age, sex, and trust fund. These projected values are passed to the Cost Module.

Awards Sample

The sample of worker beneficiaries who are newly awarded in 2019 is the foundation of the Awards Module. The sample contains a total of 52,285 newly awarded beneficiaries: 25,807 retired-worker beneficiaries and 26,478 disabled-worker beneficiaries. This sample is a subset of the 10-percent sample of newly-entitled worker beneficiaries used for the intermediate alternative.

Each record in the sample includes the worker's history of covered earnings and taxable earnings under the OASDI program, as well as additional information such as date of birth, sex, type of benefit, month of entitlement, and eligibility year of the worker. This information allows us to compute the benefits and classify each beneficiary in the sample as either a retired-worker or a disabled-worker beneficiary. Some preliminary calculations made on the sample are utilized within the model. These include the sample's *covered worker rates*, calculated separately for retired-worker and disabled-worker beneficiaries. These rates are determined using the sample's earnings histories for 1951 through 2018. They are defined for each age group, sex, and trust fund as the ratio of (1) the number of beneficiaries with covered earnings to (2) the total number of beneficiaries.

Components of the Awards Module

A goal of the Awards Module is to modify earnings histories and earnings levels in the sample to represent the earnings levels of worker beneficiaries who will be newly entitled in future years.

[38] Individuals must be age 69 or younger in order to receive a young-spouse benefit. Therefore, we exclude those who are in the age range to receive a young-spouse benefit but are receiving their own retired-worker benefit and are thus counted in the retired-worker calculation.

These modified earnings levels are used to estimate future benefit levels. The modifications include:

- **Covered Worker Rates:** The earnings histories in the sample are modified to represent the projected covered worker rate of the sample in each projected entitlement year. To do this, the covered worker rates of current and future sample cohorts are calculated using data provided by both the Economics Module and the Population Module. Then the percentage changes from the historical to the projected covered worker rates[39] are computed. The percentage changes are used to compute the number of years of earnings that need to be removed from, or added to, the sample for the projected sample earnings to meet the projected sample covered worker rates.
- **Contribution and Benefit Base:** The earnings posted for beneficiaries in the sample are limited by the historical contribution and benefit base. Prior to 1975, the maximum annual amount of earnings on which OASDI taxes were paid was determined by ad hoc legislation. After 1974, the annual maximum level was legislated to be determined automatically, based on the increases in the national average wage index (AWI). Prior to these automatic contribution and benefit base increases, a relatively large portion of workers earned amounts above the contribution and benefit base. Additional legislation raising the annual maximum taxable level occurred in 1979, 1980, and 1981. In addition, the AWI used in the automatic calculation of the annual taxable maximum was modified in the early 1990s to include deferred compensation amounts. Hence, an adjustment must be made to incorporate earnings above the historical contribution and benefit base in the sample to reflect the taxable earnings levels for future samples.
- **Average Taxable Earnings:** Average taxable earnings (ATE) may grow at a different rate than the AWI. All projected earnings in the sample are adjusted to reflect the overall increase in the ATE of future cohorts as projected by the Economics Module.

The resulting adjusted earnings histories are used to compute AIMEs for future entitlements. The Awards Module calculates the distribution of the AIME values by entitlement age, sex, and projection year, and passes the results to the Cost Module.

h. Cost

The Cost Module serves two broad purposes. The first is to compute the year-by-year progress of the combined OASI and DI Trust Funds for a 75-year projection period.[40] The second is to produce the summary measures used to assess the long-range financial status of the OASDI program for the 75-year projection period.

[39] Changes in covered worker rates are determined as the ratio of the absolute difference in the rates to the potential difference in the rates.

[40] Additionally, a projection for the 76th year is required to estimate the target fund, equal to the present value of the cost in the 76th year. See appendix B.5 for more information.

Progress of Trust Funds

In order to determine the progress of the trust funds, the dollar amounts of income and cost are computed for each year. Income includes payroll tax revenue, taxation of benefits, and interest. Cost consists of scheduled benefits, administrative expenses, and the railroad retirement interchange. Once the values of income and cost have been determined for a given year, the amount of end-of-year asset reserves is determined by starting with beginning-of-year reserves, adding income and subtracting cost. Each simulation of the model projects the progress of the trust funds for 75 years. For any simulation, full scheduled benefits are assumed to be paid from the trust funds, even if the reserves of the trust funds become depleted at some point in the projection period.

Payroll Tax Revenue

The OASDI payroll tax rate for the employer and employee, each, is currently specified as 6.2 percent, resulting in a combined employer/employee tax rate of 12.4 percent. Income to the trust funds from payroll taxes is first computed by multiplying the combined payroll tax rate by the effective taxable payroll from the Economics Module. Because there is a time lag between incurring and collecting payroll taxes, an adjustment is made when computing the annual revenue from payroll taxes.

Taxation of Benefits

Income from taxation of benefits in a given year is computed by applying a ratio to the projected benefit payments scheduled for that year. This ratio is the same as in the intermediate alternative.

Interest

The nominal yield on new-issue special public-debt obligations held by the trust funds is computed by multiplying the increase in the Consumer Price Index by the increase in the real interest rate assumed for those special public-debt obligations. Both of these values are allowed to vary and are obtained from the Assumptions Module. The nominal yield on the combined OASI and DI Trust Funds is computed in the Assumptions Module as a 7-year moving average of the nominal yields on new-issue special public-debt obligations held by the trust funds.

The amount of interest on trust fund asset reserves at the beginning of the year is calculated by multiplying the nominal yield by the amount of these reserves. Additionally, interest is calculated for amounts that enter and leave the trust funds during the year. Dollar amounts of tax income, taxation of benefits, scheduled benefits, railroad retirement interchange, and administrative expenses are exposed to, or from, the point in the year at which, on average, they are received or paid out. The amount of interest on the trust funds is then obtained by multiplying the net exposed amounts by the nominal yield for the appropriate fund.

Scheduled Benefits

A key component in projecting the amount of benefit payments for a given year is the projection of average benefits of worker beneficiaries who are newly awarded and those who are in current-payment status. The primary insurance amount (PIA) formula factors are as specified by law. The two bend points for 2023 are indexed by the increase in the average wage, as supplied by the Economics Module.

The Cost Module uses a starting average benefit matrix for worker beneficiaries in current-payment status based on a 100-percent sample of the Master Beneficiary Record (MBR)[41] as of December 31, 2022. For each sex, the starting matrix of retired-worker average benefits is broken out by *age of entitlement* to benefits (62 through 70+) and by *current age* (62 through 95+). The average benefits of retired workers who have converted from disability status are classified in a separate *age of entitlement* category. Similarly, for each sex, the starting matrix of disabled-worker average benefits is broken out by *benefit duration* (0 years through 9+ years) and by *current age* (20 through 66).

The Cost Module updates this starting matrix each year of the projection. First, average benefits are computed for worker beneficiaries who are newly awarded during the year. To compute these amounts, the distributions of AIME levels by age and sex for OASI and DI are obtained from the Awards Module. The average benefit for each age and sex is computed by applying this distribution to the intervals of potential AIME dollars, and then converting to the average PIA. Benefit amounts are determined by applying adjustment factors necessitated by provisions in law (e.g., actuarial reduction factors and delayed retirement credits). To complete the update of the matrix, two adjustments are made to the benefit levels of those who were awarded benefits in a previous year. The first is increasing benefit levels due to the COLA, as received from the Economics Module. The other adjustment is to reflect earnings after entitlement and mortality being higher for beneficiaries with lower benefits. Average benefits for worker beneficiaries are determined using the updated benefit matrix and the beneficiary matrix from the RSB and DI Modules.

Average benefits of auxiliary beneficiaries are computed by using the historical trend of the level of these benefits, as a percentage of either the average PIA or monthly benefit amount (MBA) of the primary worker beneficiary. The dollar amounts of these average benefits are computed by applying these factors to the appropriate average PIA or MBA projection. Computations are made by sex and type of auxiliary beneficiary.

Finally, the aggregate value of benefits scheduled to be paid is the sum of the products of average benefits and the numbers of beneficiaries by type of benefit with an adjustment to account for benefits paid on a retroactive basis. The numbers of beneficiaries by type of benefit are from the RSB and DI Modules.

Administrative Expenses and Railroad Retirement Interchange

The projection of administrative expenses and the railroad retirement interchange uses the same methodology as in the intermediate alternative.

[41] The MBR is a complete record of beneficiaries which contains items such as the social security number, beneficiary identification code, date of birth, date of entitlement, etc.

Year-by-Year Actuarial Measures

Once the various components of income and cost have been computed for a given year, the Cost Module computes the annual cash-flow measures and the trust fund ratio for that year. The annual cash-flow measures include income rates, cost rates, and annual balances (all as a percentage of taxable payroll and as a percentage of GDP).

Long-Range Actuarial Measures

The Cost Module computes various summary measures that are used to assess the financial status of the OASDI program during the projection period. These measures summarize, on a present-value basis, the projected annual income and cost components of the combined OASI and DI Trust Funds over the 75-year period. The assumed nominal yield on trust fund asset reserves is used to discount the annual income and cost components. The summary measures include the summarized income rate, summarized cost rate, and actuarial balance, all as percentages of taxable payroll and as percentages of GDP. An additional important summary measure is the 75-year open group unfunded obligation. All of these summary measures are stored for each simulation of the trust funds. Additional details concerning the calculations of these measures are given in appendix B.

The Cost Module also stores the years during the 75-year projection period that relate to certain trust fund events. For each simulation of the trust funds, the following special years are determined, if applicable: (1) the first year trust fund reserves are depleted, (2) the first year trust fund reserves are depleted and remain depleted throughout the projection period, (3) the first projected year that cost exceeds non-interest income, (4) the first projected year that cost exceeds non-interest income and remains in excess through the projection period, and (5) the first year that cost exceeds total income.

i. Summary Results

The Summary Results Module receives data from the Assumptions, Population, and Cost Modules for each of the stochastic simulations. It then computes the estimated probability distributions of both annual data and summarized data.

Annual Data

The Summary Results Module produces the estimated distributions of projected data available on an annual, or year-by-year, basis. Such data include the annual variables that are projected in the Assumptions Module using standard time-series modeling. In the Population Module, life expectancy data are computed and sent to the Summary Results Module for analysis. In addition, annual data projected in the Cost Module and sent to the Summary Results Module for analysis include trust fund ratios, income and cost rates, and covered workers per beneficiary. For each of the stochastic simulations, the data for a given year are sorted using Heapsort (Press, et al., 2001). The probability distributions for each year are then computed. For each piece of data (e.g., the total fertility rate), the Summary Results Module computes the smoothed empirical estimates (Klugman, et al., 1998) of the 2.5th, 5th, 10th, 20th, 30th, 40th, 50th, 60th, 70th, 80th, 90th, 95th, and 97.5th percentiles. The Summary Results Module also computes various 75-year and final 50-year averages (arithmetic or geometric, as appropriate) for each piece of annual data.

Summarized Data

Some data are only available as a summary measure for each of the stochastic simulations. Examples are the actuarial balance, summarized income and cost rates, the open group unfunded obligation, and the depletion year of the combined OASI and DI Trust Fund reserves. The Summary Results Module processes these data, most of which are passed to the module at the end of each run. At the conclusion of all of the stochastic simulations, the module computes the estimated frequency distributions for each piece of data. It computes the mean, median, and bounds for 95-, 90-, 80-, 60-, 40-, and 20-percent frequency intervals.

B. PRINCIPAL MEASURES OF FINANCIAL STATUS

Three types of financial measures are useful in assessing the actuarial status of the combined OASI and DI Trust Funds under the financing approach specified in current law: (1) annual cash-flow measures, including income rates, cost rates, and annual balances; (2) trust fund ratios; and (3) summary measures such as actuarial balances and open group unfunded obligations. In assessing the financial condition of the program, particular attention should be paid to the level of the annual balance at the end of the long-range period, and the time at which the annual balance may change from positive to negative. The ratio of beneficiaries to covered workers on a year-by-year basis (a program ratio rather than a financial ratio), provides a useful comparison to the cost of the OASDI program over the entire 75-year valuation period.

Also important is the *pattern* of projected trust fund ratios. The trust fund ratio is the proportion of a year's projected cost that can be paid with the funds available at the beginning of the year. Particular attention should be paid to the amount and year of maximum trust fund ratio, to the trust fund reserve depletion year (which is the year prior to which the trust fund ratio becomes negative), and to the stability of the trust fund ratio in cases where the ratio remains positive at the end of the long-range period.

The final measures discussed in this appendix summarize the total income and cost over valuation periods that extend through 75 years. These measures indicate whether projected income will be adequate for the period as a whole. The first such measure, the actuarial balance, indicates the size of any shortfall (when negative) as a percentage of the taxable payroll over the period. The second, the open group unfunded obligation, indicates the size of any shortfall in present-value discounted dollars.

If the 75-year actuarial balance is zero or positive, then the trust fund ratio at the end of the period, by definition, will be at least 100 percent, and financing for the program is considered to be adequate for the 75-year period as a whole. (Financial adequacy for each year is determined by whether the trust fund is non-negative throughout the year.) Whether or not financial adequacy is stable in the sense that it is likely to continue for subsequent 75-year periods in succeeding reports is also important when considering the actuarial status of the program. One indication of this stability is the behavior of the trust fund ratio at the end of the projection period. If projected trust fund ratios for the last several years of the long-range period are positive and are also stable, or rising, then it is likely that subsequent Trustees Reports will also show projections of financial adequacy (assuming no changes in demographic, economic, and programmatic assumptions).

The remaining portion of this appendix gives an overview of the trust fund operations and discusses these basic measures of assessing the actuarial status of the combined OASI and DI Trust Funds. Intermediate values from the 2023 deterministic model are often given to aid in the discussion.

1. Trust Fund Operations

Starting with the dollar level of asset reserves in a given year, the combined OASI and DI Trust Funds receive income from payroll taxes, income from taxation of benefits, and interest income.

In turn, scheduled benefits, administrative expenses, and the net financial interchange to the Railroad Retirement Board are all paid from the combined OASI and DI Trust Funds. Hence, the dollar value of reserves at the end of a given year is equal to the dollar value of reserves at the beginning of the year, plus payroll taxes, plus income from taxation of benefits, plus interest income, less scheduled benefits, less administrative expenses, less net financial interchange to the Railroad Retirement Board.

2. Annual Income Rates, Cost Rates, and Balances

Basic to the consideration of the long-range actuarial status of the trust funds are the concepts of income rate and cost rate, each of which is expressed as a percentage of taxable payroll. The annual income rate is the ratio of income from revenues (payroll tax contributions and income from the taxation of benefits) to the OASDI taxable payroll for the year. The OASDI taxable payroll consists of the total earnings which are subject to OASDI taxes, with some relatively small adjustments. These adjustments include reflecting that individuals who work for more than one employer and have OASDI covered wages exceeding the contribution and benefit base during the year will be reimbursed for the payroll taxes they paid on wages in excess of the contribution and benefit base for that year. However, each employer pays, on behalf of each employee, payroll taxes on annual wages up to the contribution and benefit base, regardless of the amount of wages an employee receives from other employers. Because the taxable payroll reflects such adjustments as these, the annual income rate is defined as the sum of the OASDI combined employee-employer contribution rate (or the payroll tax rate) scheduled in the law and the rate of income from taxation of benefits (which is, in turn, expressed as a percentage of taxable payroll). As such, it excludes net investment income. The annual cost rate is the ratio of the cost (or expenditures) of the program to the taxable payroll for the year. In this context, the cost is defined to include benefit payments, administrative expenses, and net transfers from the trust funds to the Railroad Retirement program under the financial-interchange provisions. For any year, the income rate minus the cost rate is referred to as the annual balance for the year. Social Security's cost as a percentage of the total U.S. economy (i.e., as a percentage of GDP) provides an additional perspective.

The year-by-year relationship of the income rates and cost rates illustrates the expected pattern of cash flow for the OASDI program over the full 75-year period. The pattern of the OASDI program's estimated cost rate is generally much different than that of the income rate. The income rate generally increases only slightly during the next 75 years, from a little under 13.0 percent to a little under 13.5 percent, as income from taxation of benefits slowly increases. Under the intermediate alternative, the OASDI cost rate is estimated to exceed the income rate in every year of the projection period, as it has since 2010. The cost rate is projected to rise rapidly between 2023 and 2035 as the baby-boom generation continues to reach retirement age. The projected continued reductions in death rates and relatively low birth rates will cause a significant upward shift in the average age of the population and will push the cost rate to about 18 percent of taxable payroll under the intermediate alternative. Annual deficits generally increase throughout the remainder of the 75-year projection period, reaching almost 5.1 percent of taxable payroll in 2078 and decreasing slightly to 4.3 percent by 2097.

3. Comparison of Workers to Beneficiaries

The primary reason that the estimated OASDI cost rate generally increases rapidly after 2023 is that the number of beneficiaries is projected to increase more rapidly than the number of covered workers. This occurs because the relatively large number of persons born during the baby-boom generation will reach retirement age, and begin to receive benefits, while the relatively small number of persons born during the subsequent period of low fertility rates will comprise the labor force. Based on the intermediate alternative, the number of covered workers per OASDI beneficiary declines from 2.7 in 2020 to 2.3 by 2040, and to 2.1 by 2097.

The impact of the demographic shifts on the OASDI cost rates can also be illustrated by considering the projected number of beneficiaries per 100 workers. As compared to the 2020 level of 37 beneficiaries per 100 covered workers, this ratio is estimated to be 47 in 2097 under the intermediate alternative. The trend of the OASDI cost rates is essentially the same as the trend of the number of beneficiaries per 100 covered workers over the 75-year projection period. This comparison of the trends emphasizes the extent to which the cost of the OASDI program as a percentage of taxable payroll is determined by the age distribution of the population. Because the cost rate is basically the product of the number of beneficiaries and their average benefit divided by the product of the number of covered workers and their average taxable earnings (and because average benefits rise at about the same rate as average earnings), it is to be expected that the pattern of the annual cost rates is similar to that of the annual ratios of beneficiaries to workers.

4. Trust Fund Ratios

Trust fund ratios are critical indicators of the adequacy of the financial resources of the Social Security program. The combined OASDI trust fund ratio for a given year is, by definition, the amount of asset reserves at the beginning of the year expressed as a percentage of cost for the year. In general, a yearly trust fund ratio of 100 percent or greater is desired. For any year in which the projected trust fund ratio is positive (i.e., the trust fund holds reserves at the beginning of the year) but is not positive for the following year, the trust fund is projected to become depleted during the year. Under current law, the OASI and DI Trust Funds do not have the authority to borrow, other than in the form of advance tax transfers. Therefore, depletion of the reserves in either fund during a year would mean that there are no longer sufficient funds to cover the full amount of benefits scheduled under current law.

The trust fund ratio serves an additional important purpose in assessing the actuarial status of the program. If the projected trust fund ratio is positive throughout the long-range period and is either level or increasing at the end of the period, then it is likely that subsequent reports will continue to show projected financial adequacy for the long-range period. Under these conditions, the program has achieved sustainable solvency.

The OASDI program experienced negative cash flows in 2021 for the first time since 1982, and projected income, including interest, is projected to remain lower than cost. Under the intermediate alternative, the trust fund ratio for the combined OASI and DI Trust Funds declines from 204 percent at the beginning of 2023 until the combined funds become depleted in 2034.

5. Actuarial Balance, Summarized Rates, Open Group Unfunded Obligation

The deterministic model projects nominal dollar levels of trust fund reserves over a 75-year period. In order to assess the financial status of the OASDI program quantitatively, in particular, to view the program as a whole at a specified point in time, the notion of present value, or time-value of money, is required. An essential ingredient in computing present values is the interest rate. Under the intermediate alternative, the ultimate value of the annual nominal yield on the combined OASI and DI Trust Funds is 4.76 percent, achieved in 2046. In the short-range period, projected interest earned by new special issue securities, along with interest earned on the existing reserves of the trust funds, determine implicitly the nominal yield earned by the individual OASI and DI Trust Funds in years 2023-32. The annual nominal yield of the combined OASI and DI Trust Funds, calculated through a weighting process of the two individual funds, reaches the ultimate value in 2046 under the intermediate alternative. The interest rate on the combined funds is used for present value purposes.

The 75-year long-range actuarial balance (henceforth referred to as the actuarial balance) is a measure, as a single number, of the program's financial status for the 75-year valuation period as a whole. It is essentially the difference between the income and cost of the program expressed as a percentage of taxable payroll summarized over the valuation period. When the actuarial balance is negative, the magnitude of the negative actuarial balance, the actuarial deficit, can be interpreted as the percentage that would have to be added to the current law income rate in each of the next 75 years, or subtracted from the cost rate in each year, to bring the funds into actuarial balance. Under the intermediate alternative, there is an actuarial deficit of 3.61 percent of taxable payroll for the combined OASI and DI Trust Funds.

Summarized values for the full 75-year period are useful in analyzing the long-range adequacy of financing for the program over the period as a whole under current law. The summarized income rate for the 75-year valuation period is equal to the present value of tax income, plus the beginning-of-year 2023 reserves, expressed as a percentage of the present value of taxable payroll. Similarly, the summarized cost rate for the 75-year valuation period is equal to the present value of the cost for the 75-year period, plus the present value of the 76th year's cost (the so-called "target fund"), expressed as a percentage of the present value of taxable payroll. These summarized rates are useful for comparing the total cash flows of tax income and expenditures, as an indicator of the degree to which accumulated reserves and tax income during the period are sufficient to meet the cost estimated for the long-range period. The OASDI long-range actuarial balance is defined to be the difference between the summarized 75-year income rate and the summarized 75-year cost rate.

The summarized income rate and summarized cost rate, as described above, are expressed as a percentage of the present value of taxable payroll. However, it is also useful to provide summary measures expressed in terms of the total economy of the United States, or GDP. Specifically, summary values for the OASDI program can be expressed as a percentage of the present value of GDP (instead of payroll). The summarized long-range (75-year) balance as a percentage of GDP for the OASDI program is estimated to be -1.27 percent of GDP under the intermediate alternative.

Conceptually, the open group unfunded obligation is the present value of the cash deficits observed over the 75-year period less the accumulated trust fund reserves at the beginning of the period. In particular, it is the present value of the cost of the program, less the sum of the present value of tax income and the reserves on hand at the beginning of the 75-year valuation period. Under the intermediate alternative, the open group unfunded obligation was estimated to be $22.4 trillion.

C. TIME-SERIES ANALYSIS

Time-Series Modeling

Time-series analysis is a standard set of projection techniques in econometric modeling. The equations used in the OSM to project the assumptions are fitted using these techniques. This appendix provides details about the models in general, presents statistical methods employed to test these models, and underscores the nuances inherent in the method of determining the equations. The reader may wish to consult Box, Jenkins, and Reinsel (1994), or Hamilton (1994), for a standard presentation of this material. A more elementary reference is Pindyck and Rubinfeld (1998).

Stationary Time Series

A time series Y_t is *covariance stationary* if neither the mean nor variance depends on time t:

- Y_t has a constant mean μ.
- For fixed k and all t, the covariance of Y_t and Y_{t-k} is γ_k, a constant depending on k.

In particular, the variance of Y_t is always equal to $\sigma^2 = \gamma_0$. A different concept is that of *strict stationarity*. This means that any k values of the time series have the same joint distribution as any other set of k values of the time series. If a time series Y_t is strictly stationary, then it is stationary. Henceforth, when we say that a time series Y_t is stationary we mean that it is strictly stationary. However, it is often more intuitively helpful to think of a time series as stationary in the simpler sense.

Stationarity is an important assumption in many estimation methods and approaches, because it is typically desirable that the estimated parameters depend only on the data values and not on the time at which the sample begins or ends.

Gaussian White Noise Process

Suppose that ε_t, the error term at time t, is normally distributed with mean zero and constant variance σ_ε^2. If ε_r and ε_s are uncorrelated for $r \neq s$, then the series ε_t is called a *Gaussian white noise process*. A Gaussian white noise process is stationary.

Random Walk

The simplest of all time-series models is the random walk. Here, if Y_t is the series to be estimated, then the random walk process is given by $Y_t = Y_{t-1} + \varepsilon_t$. The error term series, ε_t, is a white noise process. The forecast error variance increases as a linear function of the forecast lead time, l. In other words, the variance of the l-step-ahead forecast error is $l\sigma_\varepsilon^2$. The uncertainty spread in these forecasts will therefore widen as l increases. In other words, a random walk is not stationary.

Moving Average (MA) Models

A time series is called a moving average model of order q, or simply an MA(q) process, if

$$Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q}.$$

In this equation, $\theta_1, \dots, \theta_q$ are the moving average parameters and μ is the process mean. The error term series, ε_t, is a white noise process. An MA(q) process is always stationary for a finite q.

Autoregressive (AR) Models and Checking for Stationarity

A time series is called an autoregressive model of order p, or simply an AR(p) process, if

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t.$$

In this equation, $\phi_1, \dots \phi_p$ are the autoregressive parameters and ϕ_0 is a drift term. The naming of this model is apt because the coefficients of the time-series equation are obtained by regressing the equation on itself or, more precisely, on its own p lagged values. The error term series, ε_t, is a white noise process. If this time series is stationary, then the mean μ of this process is computed to be

$$\mu = \frac{\phi_0}{1 - \phi_1 - \phi_2 - \dots - \phi_p}.$$

There are two ways to determine whether an AR(p) process is stationary. One way involves finding the characteristic roots of the characteristic equation. This involves re-writing the AR(p) process as a lag polynomial:

$$\left(1 - \phi_1 L - \phi_2 L^2 - \cdots \phi_p L^p\right) y_t = \varepsilon_t$$

and substituting the lag operator L with a complex variable z and setting the polynomial to be equal to zero:

$$1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0.$$

The AR(p) process is stationary if all the values of z that solve the equation above lie outside the unit circle.

The alternative (and equivalent) way is to re-write the AR(p) process in matrix notation and determine whether all the eigenvalues of the $p \times p$ matrix lie inside the unit circle. For example, the matrix notation for an AR(2) process would be:

$$\begin{bmatrix} Y_t \\ Y_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_0 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ Y_{t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix}$$

Defining A as the 2×2 matrix in the equation above, λ is an eigenvalue of A and x is an eigenvector if:

$$Ax = \lambda x \Rightarrow (A - \lambda I_2)x = 0$$

$$=> A - \lambda I_p \text{ is not invertible } => \det(A - \lambda I_2) = 0$$

where

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} \phi_1 - \lambda & \phi_2 \\ 1 & -\lambda \end{bmatrix}\right)$$

$$= \lambda^2 - \phi_1 \lambda - \phi_2 = 0.$$

Solving the above equation for λ:

$$\lambda_i = \frac{\left(\phi_1 \pm \sqrt{(\phi_1^2 + 4\phi_2)}\right)}{2} \quad i = 1,2$$

If all the eigenvalues (λ_1 and λ_2 in this example) lie inside the unit circle, then the AR(2) process is stationary.

Autoregressive Moving Average (ARMA) Models

As its name indicates, an autoregressive moving average model of order (*p,q*), or simply an ARMA(*p,q*) model, is a natural combination of an autoregressive and moving average model. The equation which specifies an ARMA(*p,q*) model takes the form

$$Y_t = \mu_\phi + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}.$$

An ARMA(*p,q*) series is stationary if and only if its autoregressive part is stationary.

Deviations Form of ARMA Model Equations

It is often more convenient to transform an ARMA model equation into the deviation form using the equation

$$y_t = Y_t - \mu,$$

where μ is defined, as above, to be the mean of the process. The transformed model may be written as

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

and has a mean of zero.

Adding the original process mean to both sides of the equation produces

$$Y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}.$$

The lagged variables are left in deviations form, and the constant term, μ, on the right-hand side is the process mean.

Cholesky Decomposition

Suppose that $\vec{Z} = (Z_1, \ldots, Z_n)'$ is a vector of independently and identically distributed standard normal random variables. (If A is a matrix, then A' denotes its transpose.) Suppose we want to use these random variables to obtain a random vector $\vec{X} = (X_1, \ldots, X_n)'$ from a multivariate normal distribution with mean $\vec{\mu} = (\mu_1, \ldots, \mu_n)'$ and an $n \times n$ variance-covariance matrix $\mathbf{V} = (\sigma_{ij})$ [with $\sigma_{ij} = Cov(X_i, X_j)$ for $i, j = 1, \ldots, n$]. Since **V** is positive definite and symmetric, a standard result in linear algebra yields a lower triangular matrix, **L**, such that $\mathbf{V} = \mathbf{LL}'$. The random vector $X = \vec{\mu} + \mathbf{L}\vec{Z}$ is then a random vector with the desired properties. The decomposition $\mathbf{V} = \mathbf{LL}'$ is called a Cholesky decomposition; see Atkinson (1989) for more details. In what follows we call the matrix **L** a Cholesky matrix.

For our applications, a Cholesky decomposition is used to convert a random vector, $\vec{\varepsilon}$, of independent standard normal variates to a multivariate normal distribution with mean $\vec{\mu} = \vec{0}$ (the zero vector) and a variance-covariance matrix **V** obtained by using historical data. If **L** is a lower triangular Cholesky matrix associated with **V**, then the vector $\mathbf{L}\vec{\varepsilon}$ has the required multivariate normal distribution.

Vector Autoregressive Models

Vector autoregressive models allow the joint modeling of time-series processes. For the sake of simplicity, suppose that three variables $y_{1,t}$, $y_{2,t}$, and $y_{3,t}$ depend on time *t*. Data may indicate that these variables may be related to each other's past values. The simplest such case is when the relationship is limited to a 1-year lag, i.e., when $y_{1,t}$, $y_{2,t}$ and $y_{3,t}$ may be modeled in terms of $y_{1,t-1}$, $y_{2,t-1}$, and $y_{3,t-1}$. In this case, a three-variable VAR(1) model takes the form

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \end{bmatrix} = B \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{3,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}$$

for some 3 x 3 matrix *B*.

Alternatively stated, if $\vec{y}_t = (y_{1,t}, y_{2,t}, y_{3,t})'$ and $\vec{\varepsilon}_t = (\varepsilon_{1,t}, \varepsilon_{2,t}, \varepsilon_{3,t})'$ then the model takes the form $\vec{y}_t = B\vec{y}_{t-1} + \vec{\varepsilon}_t$ for some 3×3 matrix *B*.

The *k*-variable VAR(p) model, with p lags, naturally extends from this. If $\vec{y}_t = (y_{1,t}, y_{2,t}, \ldots, y_{k,t})'$ and $\vec{\varepsilon}_t = (\varepsilon_{1,t}, \varepsilon_{2,t}, \ldots, \varepsilon_{k,t})'$ then the *k*-variable VAR(p) model takes the form $\vec{y}_t = B_1\vec{y}_{t-1} + B_2\vec{y}_{t-2} + \ldots + B_p\vec{y}_{t-p} + \vec{\varepsilon}_t$ for $k \times k$ matrices $B_1, \ldots, B_p$.

Adapting the Vector Autoregressive Model to Parameter Uncertainty

This section describes the procedure to obtain parameters adjusted for parameter uncertainty for the three-equation vector autoregressive model with two lags that was used to model the unemployment rate U_t, the inflation rate I_t, and the real interest rate R_t.

The three-equation VAR with two lags can be written in matrix notation as:

$$\begin{bmatrix} U_t \\ I_t \\ R_t \end{bmatrix} = \begin{bmatrix} \mu_U \\ \mu_I \\ \mu_R \end{bmatrix} + \mathrm{A}_1 \begin{bmatrix} U_{t-1} - \mu_U \\ I_{t-1} - \mu_I \\ R_{t-1} - \mu_R \end{bmatrix} + \mathrm{A}_2 \begin{bmatrix} U_{t-2} - \mu_U \\ I_{t-2} - \mu_I \\ R_{t-2} - \mu_R \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}$$

where

$$\mathrm{A}_1 = \begin{bmatrix} a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} \end{bmatrix}$$

$$\mathrm{A}_2 = \begin{bmatrix} a_{112} & a_{122} & a_{132} \\ a_{212} & a_{222} & a_{232} \\ a_{312} & a_{322} & a_{332} \end{bmatrix}$$

As there are 21 parameters to estimate (μ_U, μ_I, μ_R, and the parameters in matrices A_1 and A_2), there will be 21 parameters that need adjustments for full parameter uncertainty. When applying parameter uncertainty only around the expected means, only the adjustments on the three mean terms are used and the adjustments to the constants in the A_1 and A_2 matrices are ignored.

For full parameter uncertainty, the lower triangular Cholesky matrix of the 21 x 21 parameter variance-covariance matrix is multiplied by a 21 x 1 vector of standard normal random numbers to obtain a 21 x 1 vector of adjustments which are added to the estimated parameters to produce a final set of parameters.

To verify that the parameters with full parameter uncertainty pass the stationarity test, the maximum eigenvalue of the following matrix is calculated:

$$\begin{bmatrix} \tilde{a}_{111} & \tilde{a}_{121} & \tilde{a}_{131} & \tilde{a}_{112} & \tilde{a}_{122} & \tilde{a}_{132} \\ \tilde{a}_{211} & \tilde{a}_{221} & \tilde{a}_{231} & \tilde{a}_{212} & \tilde{a}_{222} & \tilde{a}_{232} \\ \tilde{a}_{311} & \tilde{a}_{321} & \tilde{a}_{331} & \tilde{a}_{312} & \tilde{a}_{322} & \tilde{a}_{332} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

In the matrix above, the $\tilde{a}_{ijk}$ terms denote the autoregressive parameters adjusted for parameter uncertainty. If the maximum eigenvalue of the matrix lies within the unit circle, then the parameters pass the stationarity test and are retained for stochastic simulation. However, when the model is run with parameter uncertainty only around the mean terms, these stationarity tests are unnecessary.

D. MONTE CARLO SIMULATION

A standard reference for this appendix is Knuth (1997). In choosing a random number generator, it is essential to consider the following:

- the generator should have the correct statistical properties
- the stream should be easy to reproduce
- the cycle length should be long
- computational efficiency and storage needs should be met
- the generator should be portable across platforms
- it should be easy to generate separate streams

Given a seed I_0 and a judicious choice of a and m, the linear congruence $I_{j+1} = aI_j (mod\ m)$ will generate a sequence of numbers. The values $a = 16{,}807$ and $m = 2^{31} - 1$ have the property that, for any nonzero initial seed, a large number of variables may be generated before cycling occurs. Upon division by m, a sequence of variates U_j of a uniform distribution on the interval (0,1) is generated. This linear congruential generator with these values of a and m is the so-called "Minimal Standard" generator of Park and Miller and appears in Press et al. (2001). As explained there, the value of 0 must never be allowed as the initial seed. Also, since a and m are relatively prime, the value of 0 will never occur provided that the initial seed is nonzero. This choice of a and m gives a cycle of maximum possible length $2^{31} - 2$. The Minimal Standard generator can be modified so as to remove low-order serial correlations inherent in the random numbers produced by that method. The method is adapted from Press et al. (2001) and uses the Park-Miller algorithm along with a Bays-Durham shuffle. This is the process the OSM uses to generate uniform (0,1) variates. There are other methods which will further eliminate such serial correlations. We chose this particular method to balance the need to remove such serial correlations with the expense of run-time.

To generate the standard normal (mean 0 and variance 1) variates used to generate the random error terms for the equations, one may transform uniform variates using the Box-Muller method: generate two independent uniform (0,1) variates, U_1 and U_2, and compute $X = \sqrt{-2\ln U_1} \cdot \cos(2\pi U_2)$ and $Y = \sqrt{-2\ln U_1} \cdot \sin(2\pi U_2)$. In order to avoid the use of trigonometric functions, the Polar method is used. This method, as a preliminary step, transforms the two uniform (0,1) variates U_1 and U_2 to uniform (–1,1) variates V_1 and V_2 (using the transformation $V = 2U - 1$). If $S = V_1^2 + V_2^2$, then $X = V_1\sqrt{\frac{-2\ln S}{S}}$ and $Y = V_2\sqrt{\frac{-2\ln S}{S}}$ will be two independent random draws from a standard normal distribution, provided that S is less than one. (If not, then this process is repeated until it is.) Although this method produces two standard normal variates, the first one (X) is used and the second one (Y) is discarded. Each equation is separately seeded to have its own random number stream.

E. PARAMETER UNCERTAINTY IMPLEMENTATIONS

There are two ways to implement parameter uncertainty: (1) parameter uncertainty around the mean and (2) full parameter uncertainty. As mentioned in section II.B, under full parameter uncertainty, if a particular set of random numbers results in AR coefficients that are nonstationary for a particular assumption, this draw is ignored and not used as one of the 5,000 simulations. For more discussion on nonstationarity, see section II.B and appendix C. However, as a quick summary, changing the coefficients requires a stationarity recheck, because stationarity is required for the equations in the OSM. If an equation is nonstationary, summary measures can increase in magnitude without bounds, resulting in implausible scenarios. Additionally, the equations are estimated assuming stationarity and thus, nonstationary series would be inconsistent with the estimation procedure. For these reasons, stationarity is a requirement for ARMA and VAR equations in the OSM. Nonstationarity is determined as having a maximum absolute value eigenvalue of greater than or equal to 1.0. Table V.2 shows the percentage of draws that result in nonstationarity when using full parameter uncertainty.

Table V.2 – Nonstationary Draw Percentages Under Full Parameter Uncertainty

Assumption[1]	Percentage Nonstationary
Total Fertility Rate	9.1
LPR New Arrival Immigration	0.6
Legal Emigration Rate	0.0
Other-than-LPR Immigration	0.6
Transfer Rate	0.0
Economic VAR Framework[2]	6.7
Male Disability Incidence Rate	21.1
Female Disability Incidence Rate	5.1
Male Disability Recovery Rate	5.5
Female Disability Recovery Rate	6.1

[1] The 42 age-group/sex mortality assumptions are not listed. However, none of the draws in any of the simulations resulted in equations that were nonstationary. The real average covered wage is also not listed because it is based on unemployment only and therefore only the percentage of nonstationary draws for unemployment (i.e., the economic VAR framework) is relevant for this assumption.
[2] The economic vector autoregression (VAR) framework determines the unemployment rate, inflation rate, and real interest rate simultaneously.

As shown in table V.2, several assumptions have a relatively high number of nonstationary draws under full parameter uncertainty. These are the total fertility rate, economic VAR framework, and both sets of disability rates (incidence and recovery). Particularly notable is the 21.1 percent of draws that are nonstationary for the male disability incidence rate assumption. For each of these assumptions, figures are presented below that correspond to those shown in section IV.A, but also contain additional lines for full parameter uncertainty. In addition, for comparison, a figure is presented for an assumption that does not have a large number of nonstationary draws (other-than-LPR immigration).

Removing the nonstationary draws may lead to biased distributions of results. One would expect full parameter uncertainty to have a wider 95-percent frequency interval than parameter uncertainty for the expected mean in the short term, which can be seen in all of the figures below. However, removing nonstationary draws can result in a skewed distribution or excess kurtosis (more weight of the distribution in the tails relative to a normal distribution). A skewed distribution or excess kurtosis can occur because draws that would result in values at the upper or lower portion of the distribution are removed, thus biasing the distribution in the opposite direction. One reason to use parameter uncertainty for the expected mean (rather than full parameter uncertainty) is that normally distributed frequency distributions (with skew and excess kurtosis both close to zero) are desired in simulations like the OSM.

Another reason to use parameter uncertainty for the expected mean is that most of the additional variation in the resulting distributions comes from parameter uncertainty around the mean and not from parameter uncertainty of the coefficients. In fact, one can show that in the long run, full parameter uncertainty and parameter uncertainty for the expected mean have cumulative averages that converge under certain assumptions (including certain conditions of normality). This convergence is evident in figure V.1b, which shows cumulative average values for other-than-LPR immigration. Other-than-LPR immigration is an example of an assumption with a low number of nonstationary draws. However, convergence is also evident in figure V.2b, which shows cumulative average values for the total fertility rate, even with the obvious issues of skew and excess kurtosis in the annual values in figure V.2a. Convergence is not evident for the assumptions with a high number of nonstationary draws, as shown in figures V.3b, V.4b, V.5b, and V.6b. However, we know that these full parameter uncertainty distributions are potentially distorted due to the dismissal of all the nonstationary draws. Note in particular figures V.6a and V.6b show a very high distortion under full parameter uncertainty with male disability incidence rates. The fact that male disability incidence rate shows the highest distortion should not be surprising given this assumption has the highest number of nonstationary draws of all variables in table V.2.

Figure V.1a – Other-than-LPR Immigration Annual Values, with Full Parameter Uncertainty Added, Calendar Years 1999-2097

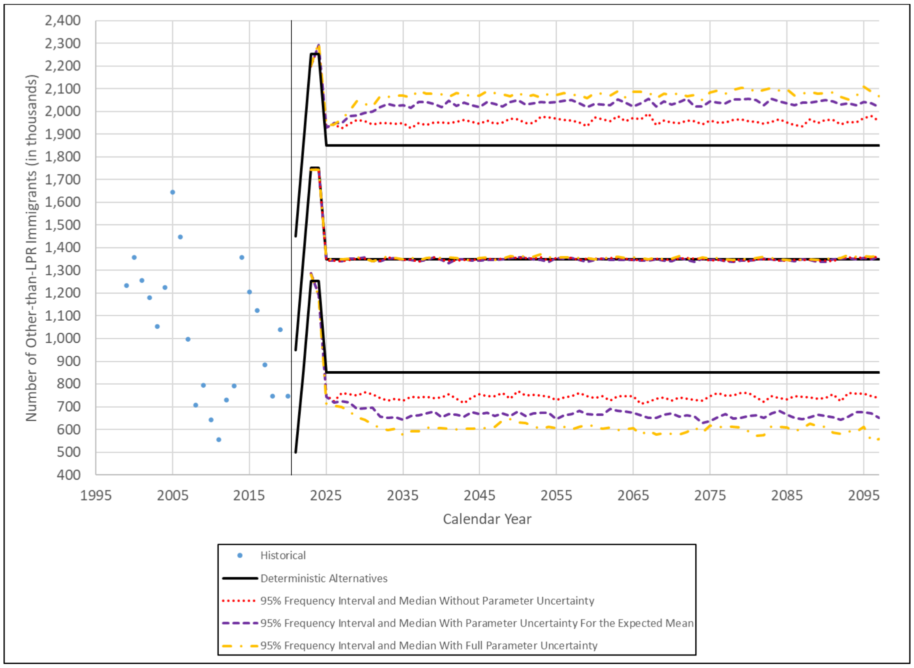


Figure V.1b – Other-than-LPR Immigration Cumulative Average Values, with Full Parameter Uncertainty Added, Calendar Years 1999-2097

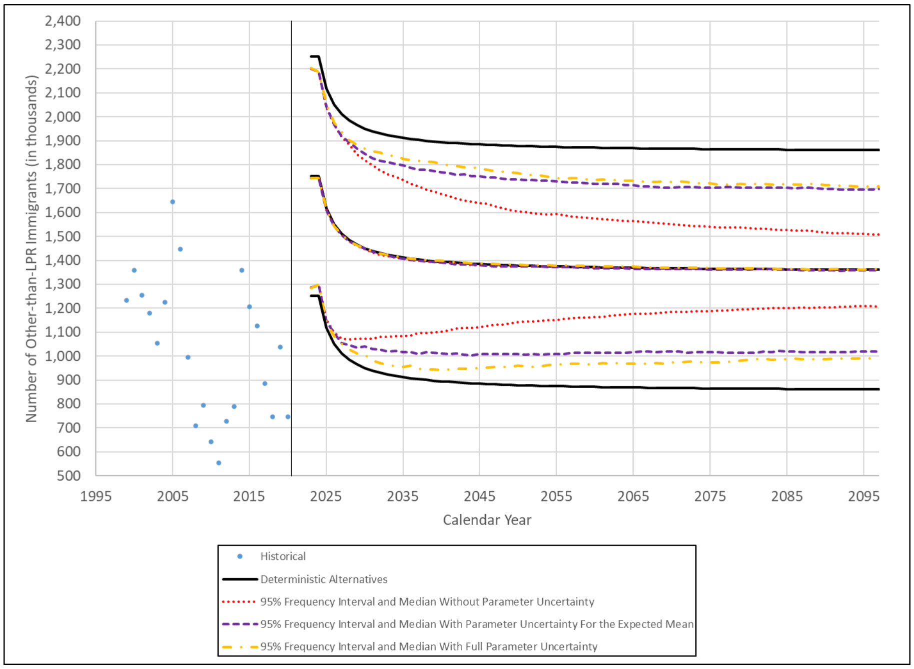


Figure V.2a – Total Fertility Rate Annual Values, with Full Parameter Uncertainty Added, Calendar Years 1917-2097

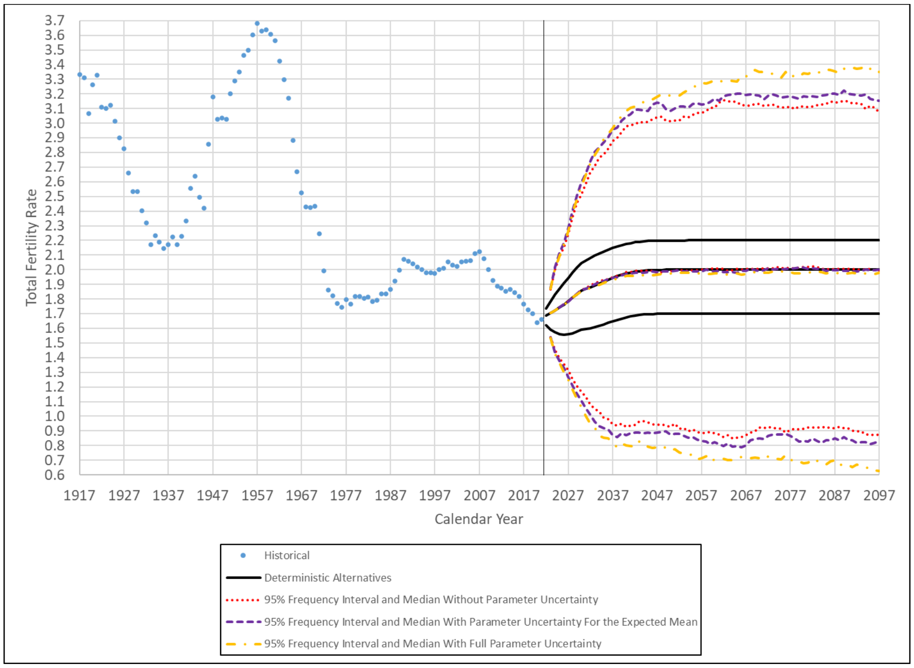


Figure V.2b – Total Fertility Rate Cumulative Average Values, with Full Parameter Uncertainty Added, Calendar Years 1917-2097

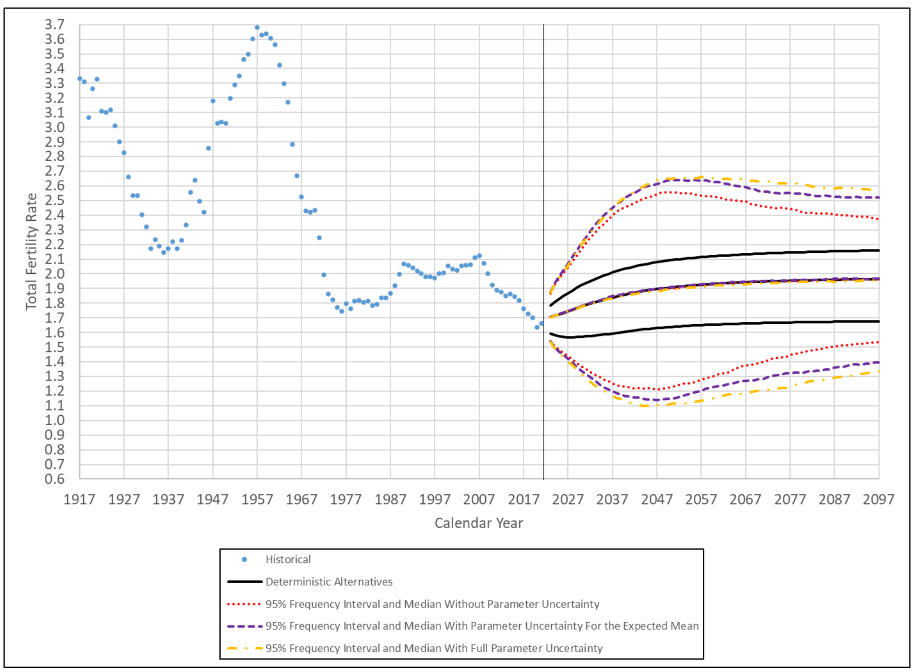


Figure V.3a – Unemployment Rate Annual Values, with Full Parameter Uncertainty Added, Calendar Years 1961-2097

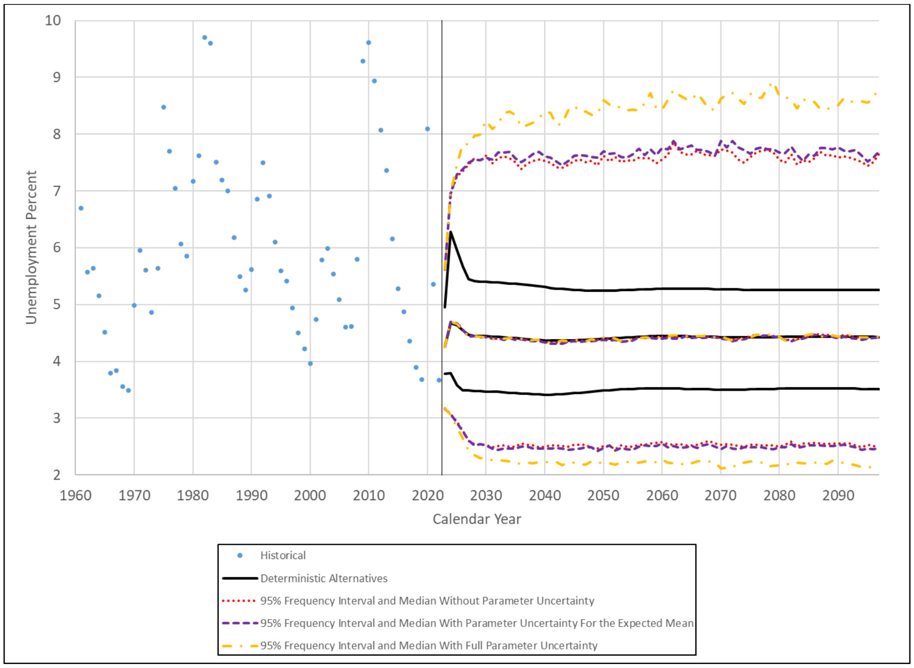


Figure V.3b – Unemployment Rate Cumulative Average Values, with Full Parameter Uncertainty Added, Calendar Years 1961-2097

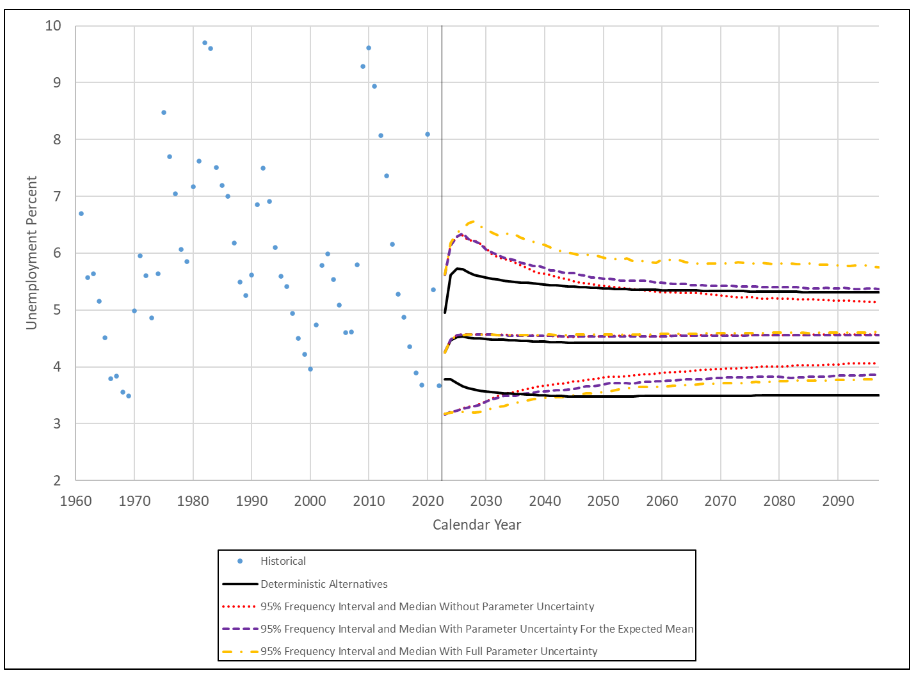


Figure V.4a – Inflation Rate Annual Values, with Full Parameter Uncertainty Added, Calendar Years 1961-2097

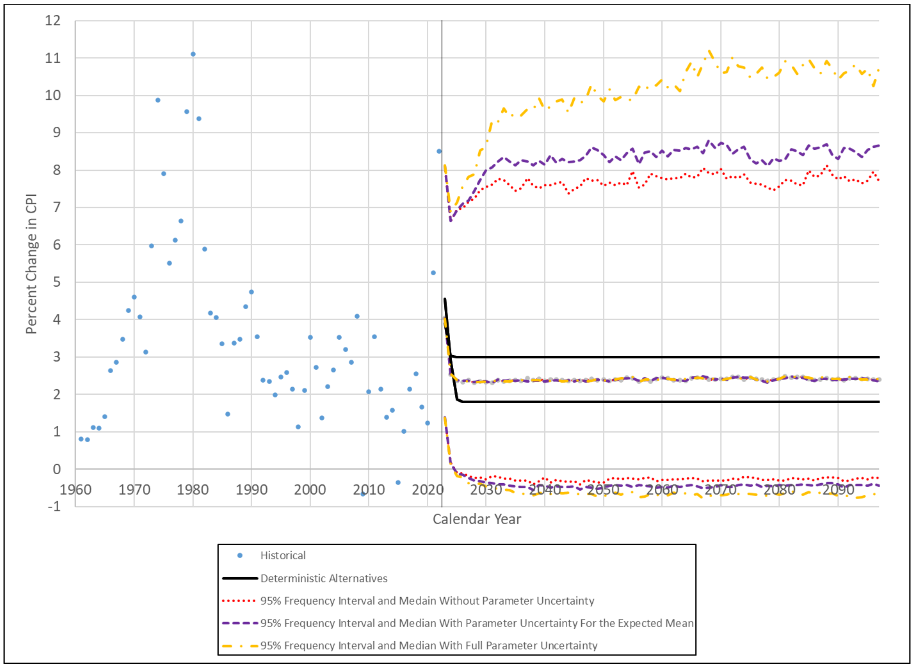


Figure V.4b – Inflation Rate Cumulative Average Values, with Full Parameter Uncertainty Added, Calendar Years 1961-2097

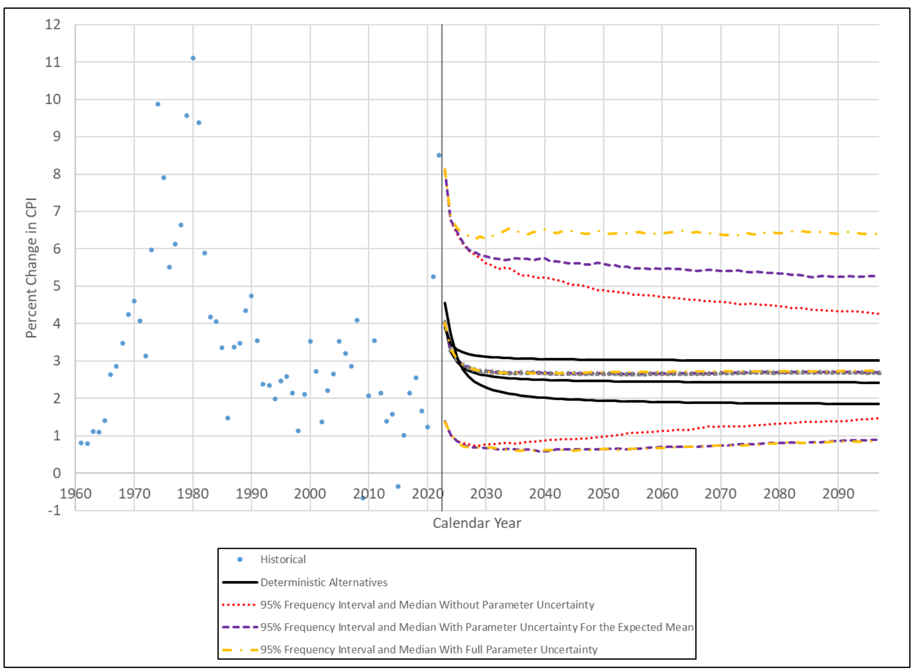


Figure V.5a – Real Interest Rate Annual Values, with Full Parameter Uncertainty Added, Calendar Years 1961-2097

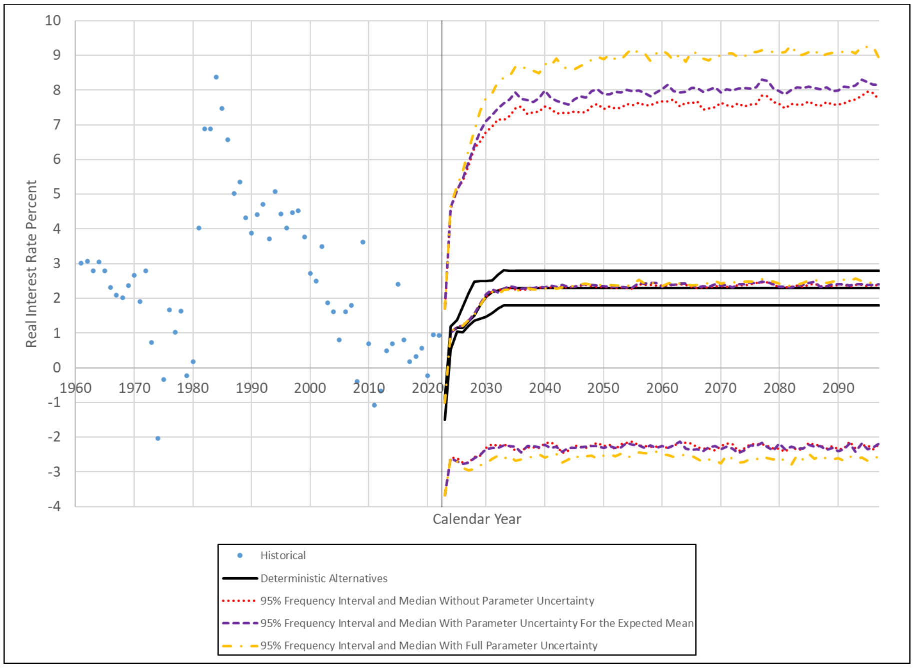


Figure V.5b – Real Interest Rate Cumulative Average Values, with Full Parameter Uncertainty Added, Calendar Years 1961-2097

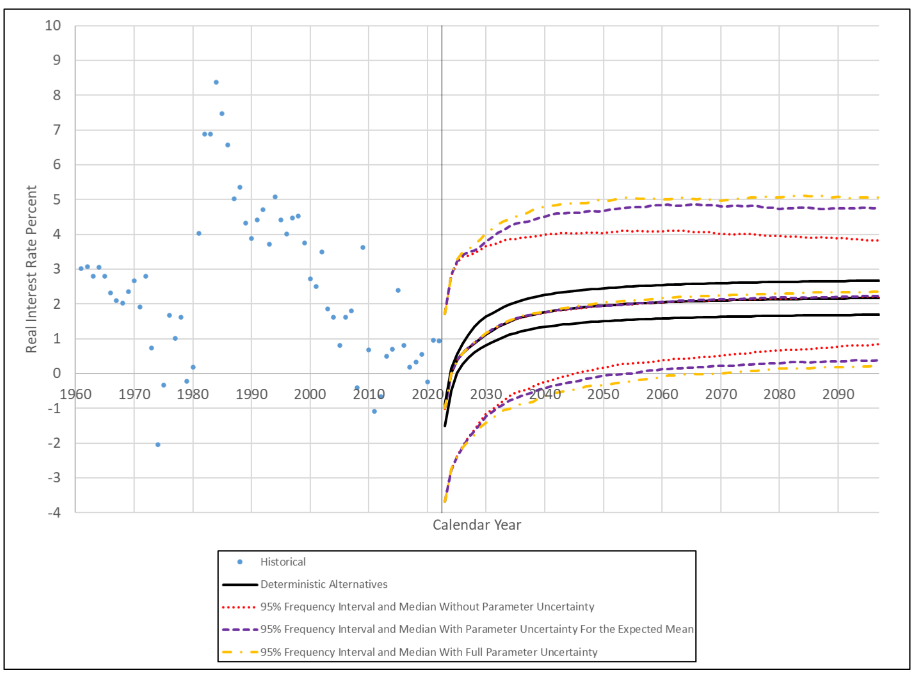


Figure V.6a – Male Disability Incidence Rate Annual Values, with Full Parameter Uncertainty Added, Calendar Years 1970-2097

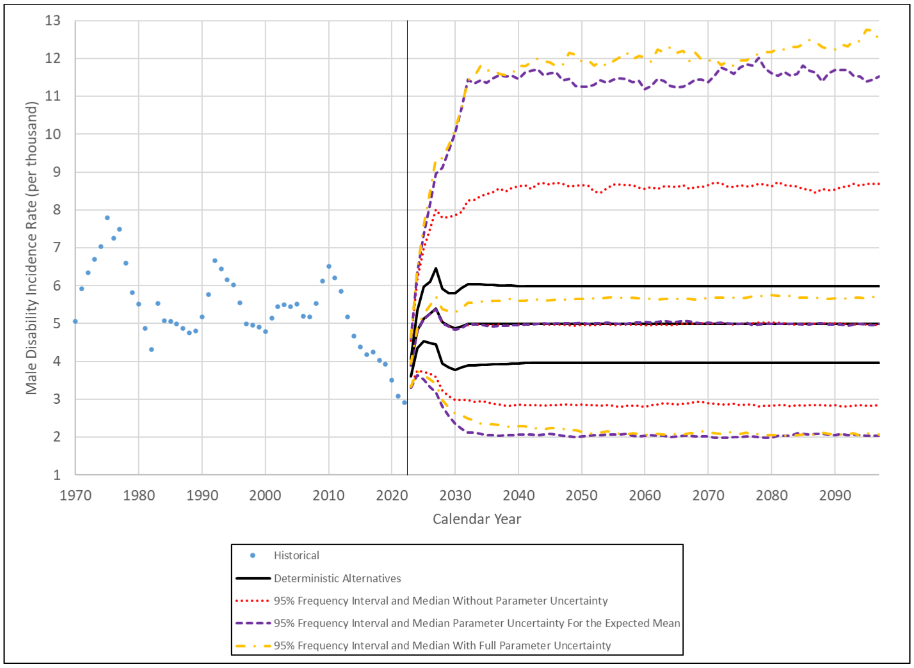
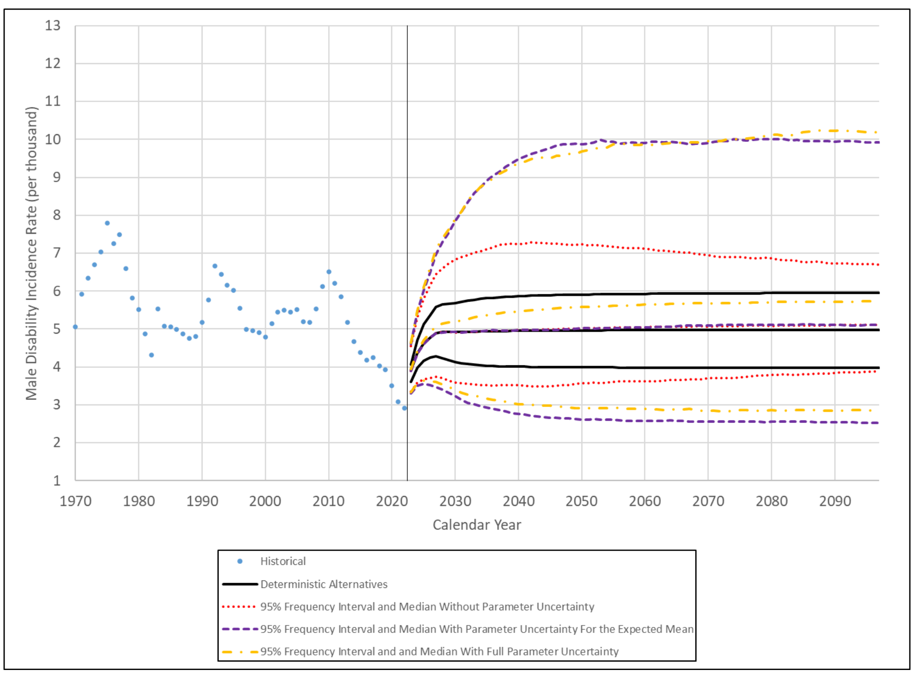


Figure V.6b – Male Disability Incidence Rate Cumulative Average Values, with Full Parameter Uncertainty Added, Calendar Years 1970-2097



The summary values for parameter uncertainty for the expected mean and full parameter uncertainty are also of interest. Below are figures showing the trust fund ratios (figure V.7) and cost rates as a percentage of taxable payroll (figure V.8) for full parameter uncertainty, parameter uncertainty for the expected mean, and no parameter uncertainty.

Figure V.7 – Annual Trust Fund (Unfunded Obligation) Ratios, Calendar Years 2023-97

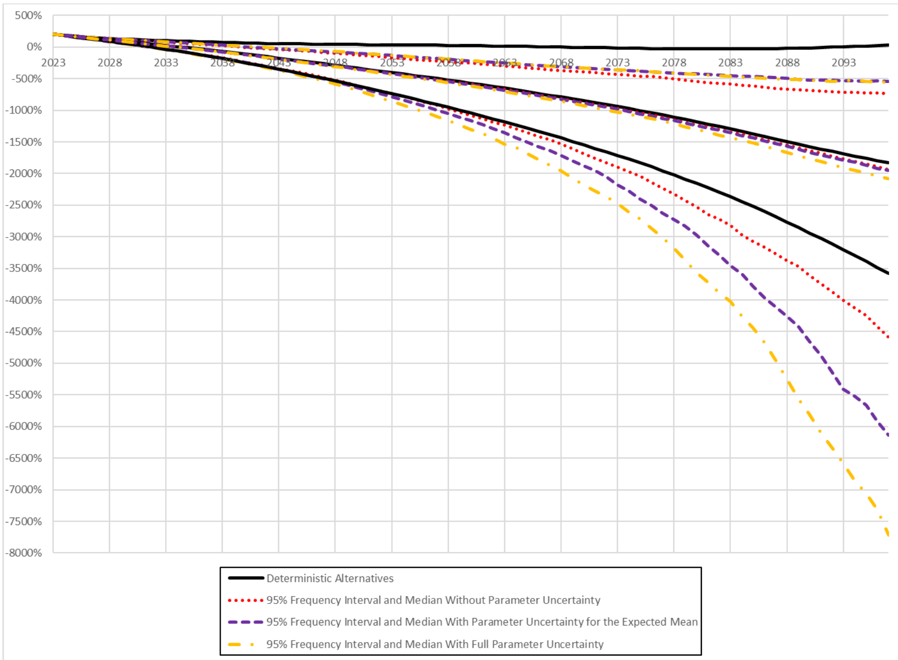
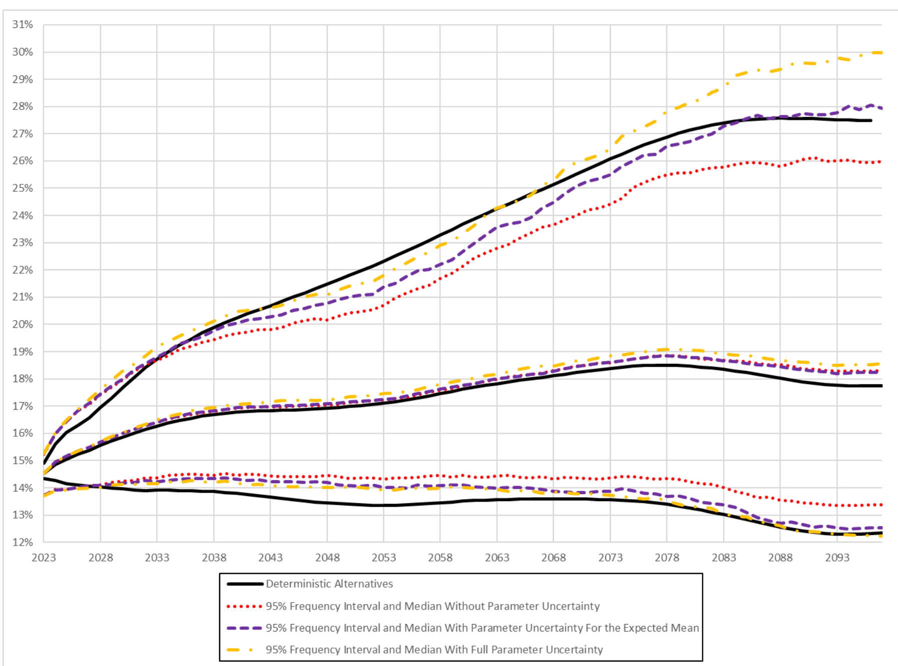


Figure V.8 – Annual Cost Rates, Calendar Years 2023-97



As expected in figures V.7 and V.8, the ranges with parameter uncertainty for the expected mean are narrower than those with full parameter uncertainty.

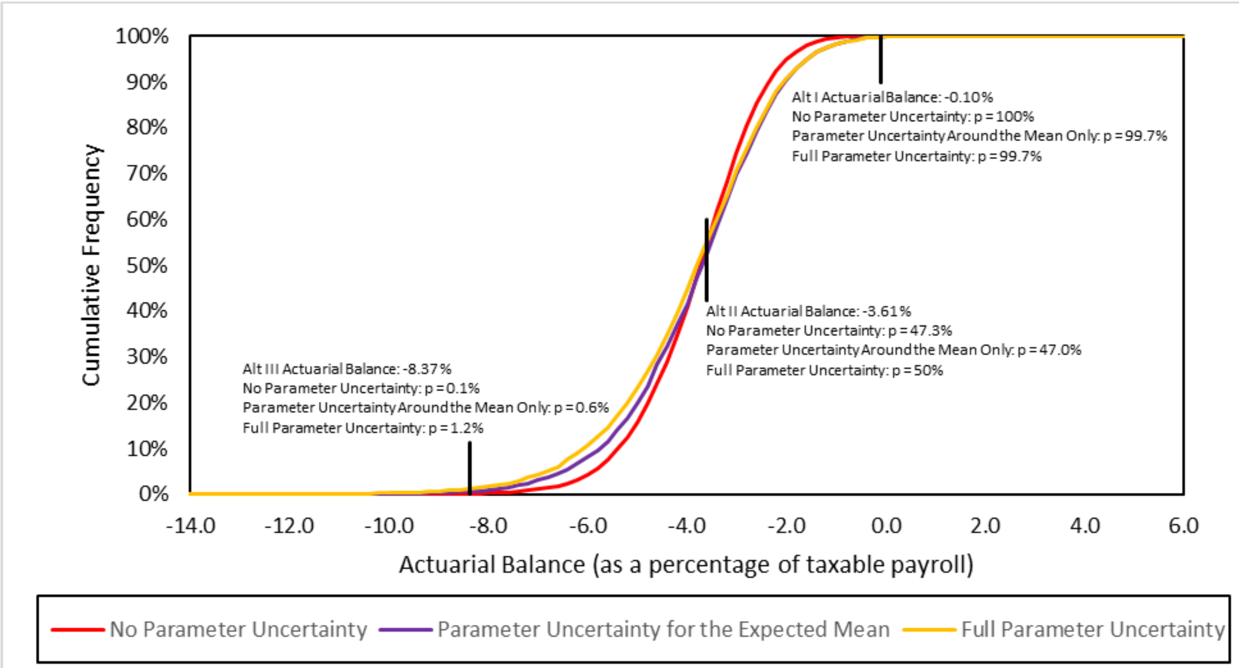
Finally, data about the actuarial balance is shown in table V.3.

Table V.3 – Actuarial Balance (as a Percentage of Taxable Payroll) in the Stochastic Model

	Simulation		
Metric	No Parameter Uncertainty	Parameter Uncertainty for the Expected Mean	Full Parameter Uncertainty
Median	-3.72	-3.69	-3.80
Average	-3.82	-3.82	-3.95
2.5th percentile	-6.39	-7.17	-7.59
97.5th percentile	-1.68	-1.18	-1.19
Standard Deviation	1.20	1.51	1.63

Looking at table V.3, the difference in the actuarial balance between the median (or average) with full parameter uncertainty and the median (or average) with no parameter uncertainty is larger than the difference between the median (or average) with parameter uncertainty for the expected mean and the median (or average) with no parameter uncertainty. Without any parameter uncertainty, the 95-percent frequency interval ranges from -6.39 percent of taxable payroll to -1.68 percent of taxable payroll, for a total range of 4.71 percent of taxable payroll. With parameter uncertainty for the expected mean, the 95-percent frequency interval is notably larger, ranging from -7.17 percent of taxable payroll to -1.18 percent of taxable payroll, for a total range of 5.99 percent of taxable ratio. Under full parameter uncertainty, the 95-percent frequency interval ranges from -7.59 percent of taxable payroll to -1.19 percent of taxable payroll, for a total range of 6.40 percent of taxable payroll. This range is not much larger than the 95-percent frequency interval with parameter uncertainty for the expected mean. Similarly, the standard deviation of the actuarial balance is 1.20 percent of taxable payroll without any parameter uncertainty but is 1.51 percent of taxable payroll with parameter uncertainty for the expected mean. Using full parameter uncertainty leads to a slightly larger standard deviation of 1.63 percent of taxable payroll. Figure V.9 shows the cumulative frequency distributions for the actuarial balance. Note that the line for full parameter uncertainty (yellow) is almost on top of the line for parameter uncertainty for the expected mean (purple).

Figure V.9 – Actuarial Balance Cumulative Percentage Frequency of 5,000 Simulations with Full Parameter Uncertainty Added



As explained above, there are complications inherent in implementing full parameter uncertainty. In particular, removing a large number of draws due to nonstationarity for certain assumptions can lead to skewed distributions and excess kurtosis. In addition, most of the effect of full parameter uncertainty is captured by having parameter uncertainty for the expected mean. Therefore, we decided to use parameter uncertainty for the expected mean for the current version of the OSM, beginning with the 2021 Trustees Report. We will continue to explore the feasibility of implementing full parameter uncertainty.

F. EQUATION REGRESSION STATISTICS, PARAMETER ESTIMATES, AND BOUNDING STATISTICS

This section of the appendix contains the specification of the equations as they were estimated as well as the estimated parameter values, standard errors, and parameter variance-covariance matrices. The parameter values include, where applicable, the estimated autoregressive (AR) and moving average (MA) coefficients. In addition, the assumed annual projected values of each variable from the 2023 Trustees Report alternative II (e.g., ${F_t}^{TR}$) are presented in a table in the second part of this appendix. For each equation presented here, there is a corresponding equation described in chapter II.

The standard errors of each regression were used to generate random variation in each of the variables. The random variation in one variable is usually generated independently of the random variation in another variable. However, the mortality improvement variables, economic VAR framework variables, disability incidence variables, and disability recovery variables were each generated in such a way that the random error terms were correlated. For each of these, a Cholesky decomposition was performed on a residual variance-covariance matrix. The correlated random error terms were then obtained by multiplying the lower triangular Cholesky matrix by a vector of standard normal random errors.

The parameter variance-covariance matrix from each estimated equation was used to introduce parameter uncertainty into the estimated parameters. For each parameter variance-covariance matrix, a Cholesky decomposition was performed, and adjustments to the estimated parameters were created by multiplying the lower triangular Cholesky matrix by a vector of standard normal random numbers. When applying parameter uncertainty for the estimated mean, as discussed in appendix E, only the adjustments to the mean term were added to the assumed annual projected values from the 2023 Trustees Report, and the other adjustments to the AR and MA parameters calculated by the multiplication above were set to zero. The adjusted means change from one simulation to the next but remain the same over the 75-year projection period for each simulation.

Some equations are bounded for reasonableness. The third part of this appendix displays the percentage of runs that are bounded.

1. Regression Statistics

Total Fertility Rate

$$F_t = \mu + \phi_1(F_{t-1} - \mu) + \phi_2(F_{t-2} - \mu) + \phi_3(F_{t-3} - \mu) + \phi_4(F_{t-4} - \mu) + \varepsilon_t + \theta_1\varepsilon_{t-1},$$

Note: Because the total fertility rate must be non-negative, its lower bound was set at zero. An upper bound for the fertility rate was set at an effectively infinite value of 100,000.

AR parameters	MA parameters	Regression Mean Over Historical Period	Standard Error
$\phi_1 = 1.961953$	$\theta_1 = -0.599494$	$\mu = 2.302764$	$\sigma_\varepsilon = 0.083837$
$\phi_2 = -1.456856$			
$\phi_3 = 0.881344$			
$\phi_4 = -0.405045$			

Cholesky decomposition of the parameter variance-covariance matrix from the estimated fertility equation, with the parameters in the following order: μ, ϕ_1, ϕ_2, ϕ_3, ϕ_4, θ_1:

$$\begin{bmatrix} 0.194726 & 0 & 0 & 0 & 0 & 0 \\ 0.068651 & 0.185479 & 0 & 0 & 0 & 0 \\ -0.152632 & -0.320726 & 0.152159 & 0 & 0 & 0 \\ 0.139752 & 0.149100 & -0.286136 & 0.049555 & 0 & 0 \\ -0.053947 & -0.009164 & 0.137069 & -0.048859 & 0.006084 & 0 \\ -0.060572 & -0.132401 & -0.037497 & -0.037215 & -0.007022 & 0.077445 \end{bmatrix}$$

Rate of Decrease in Central Death Rate

$$M_{k,t} = \mu_k + \phi_{k_1}(M_{k,t-1} - \mu_k) + \varepsilon_{k,t},$$

Group (k)	Sex	Age	AR Parameter ϕ_{k_1}	Regression Mean Over Historical Period μ_k
1	M	Under 1	-0.321102	2.649174
2	F	Under 1	-0.346321	2.614814
3	M	1 to 4	-0.415383	3.200755
4	F	1 to 4	-0.420574	3.258776
5	M	5 to 9	-0.230105	2.764799
6	F	5 to 9	-0.280705	2.767535
7	M	10 to 14	-0.231467	1.943431
8	F	10 to 14	-0.191579	2.100112
9	M	15 to 19	-0.180280	0.843267
10	F	15 to 19	-0.161958	1.716487
11	M	20 to 24	-0.165854	0.550723
12	F	20 to 24	-0.190482	1.429141
13	M	25 to 29	-0.205471	0.213847
14	F	25 to 29	-0.181090	1.030135
15	M	30 to 34	-0.202313	0.315454
16	F	30 to 34	-0.205214	1.144602
17	M	35 to 39	-0.181789	0.653697
18	F	35 to 39	-0.203269	1.268761
19	M	40 to 44	-0.082093	0.821361
20	F	40 to 44	-0.180428	1.296906
21	M	45 to 49	-0.129337	0.817384
22	F	45 to 49	-0.206066	1.164831
23	M	50 to 54	-0.071034	0.720460
24	F	50 to 54	-0.236083	1.110494
25	M	55 to 59	-0.023481	0.651642
26	F	55 to 59	-0.191735	1.006140
27	M	60 to 64	-0.076255	0.622407
28	F	60 to 64	-0.262722	1.001435
29	M	65 to 69	-0.171710	0.666208
30	F	65 to 69	-0.226507	0.990396
31	M	70 to 74	-0.255254	0.696362
32	F	70 to 74	-0.272665	0.970823
33	M	75 to 79	-0.244379	0.615691
34	F	75 to 79	-0.292959	0.856685
35	M	80 to 84	-0.327045	0.593276
36	F	80 to 84	-0.306926	0.758256
37	M	85 to 89	-0.244672	0.395173
38	F	85 to 89	-0.202911	0.508494
39	M	90 to 94	-0.258088	0.258191
40	F	90 to 94	-0.228274	0.385726
41	M	95 and Older	-0.176282	0.076520
42	F	95 and Older	-0.226741	0.247714

Cholesky decomposition of the residual variance-covariance matrix:

$$V = LL' \qquad \mathbf{L} = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \qquad (\mathbf{L} \text{ is a 42 x 42 matrix})$$

$$A_1 = \begin{bmatrix}
4.39 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4.31 & 1.09 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4.11 & 1.29 & 6.09 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4.42 & 1.95 & 5.96 & 2.77 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3.17 & 1.19 & 4.17 & 1.72 & 3.65 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3.60 & 0.71 & 5.47 & 1.97 & 3.16 & 3.29 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3.07 & 0.96 & 4.55 & 1.72 & 3.45 & 1.26 & 4.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4.12 & 0.88 & 5.98 & 1.99 & 4.23 & 2.25 & 1.91 & 3.89 & 0 & 0 & 0 & 0 & 0 & 0 \\
4.34 & 1.78 & 7.88 & 3.12 & 4.07 & 1.58 & 4.26 & 1.85 & 4.39 & 0 & 0 & 0 & 0 & 0 \\
3.71 & 1.31 & 6.61 & 2.57 & 3.69 & 1.69 & 3.24 & 2.59 & 2.09 & 2.89 & 0 & 0 & 0 & 0 \\
3.93 & 2.43 & 8.45 & 3.23 & 4.26 & 1.58 & 4.07 & 1.70 & 4.49 & -0.70 & 3.39 & 0 & 0 & 0 \\
4.62 & 1.79 & 8.12 & 3.68 & 4.22 & 2.31 & 4.25 & 2.42 & 3.21 & 2.07 & 0.41 & 3.39 & 0 & 0 \\
5.10 & 2.65 & 10.16 & 4.42 & 5.17 & 2.49 & 4.88 & 2.77 & 5.40 & 0.49 & 1.81 & 2.01 & 2.99 & 0 \\
5.08 & 2.45 & 9.92 & 4.46 & 4.86 & 2.78 & 4.76 & 3.31 & 4.08 & 2.29 & 0.18 & 3.96 & 1.63 & 2.24
\end{bmatrix}$$

$$A_2 = \begin{bmatrix}
4.92 & 2.57 & 8.85 & 3.47 & 4.32 & 2.02 & 5.12 & 2.57 & 4.24 & 0.72 & 1.16 & 1.86 & 2.25 & 0.53 \\
4.15 & 2.05 & 7.54 & 3.22 & 3.85 & 2.20 & 3.93 & 2.29 & 2.88 & 1.40 & 0.40 & 2.41 & 0.95 & 1.42 \\
3.52 & 1.50 & 5.55 & 1.92 & 2.54 & 0.78 & 3.74 & 1.41 & 2.32 & 0.71 & 0.81 & 0.20 & 1.32 & 0.33 \\
2.71 & 1.08 & 4.42 & 1.76 & 2.12 & 1.01 & 2.85 & 1.19 & 1.39 & 1.10 & 0.60 & 0.94 & 0.15 & 0.82 \\
2.43 & 0.95 & 2.90 & 0.41 & 1.07 & 0.09 & 2.52 & 0.57 & 1.22 & 0.72 & 0.81 & -1.03 & 0.12 & -0.02 \\
1.95 & 0.59 & 2.47 & 0.52 & 1.01 & 0.48 & 1.97 & 0.64 & 0.86 & 1.07 & 0.76 & -0.05 & -0.34 & 0.30 \\
1.87 & 0.40 & 1.68 & -0.20 & 0.46 & -0.20 & 1.94 & 0.32 & 0.80 & 0.53 & 0.53 & -0.88 & -0.06 & 0.10 \\
1.53 & 0.25 & 1.81 & -0.02 & 0.49 & 0.29 & 1.43 & 0.27 & 0.43 & 0.83 & 0.39 & -0.12 & -0.32 & 0.36 \\
1.39 & 0.22 & 0.89 & -0.42 & 0.03 & -0.49 & 1.27 & -0.07 & 0.51 & 0.65 & 0.66 & -0.91 & -0.23 & 0.24 \\
1.08 & 0.44 & 1.49 & -0.20 & 0.38 & 0.01 & 1.11 & 0.04 &]0.13 & 0.77 & 0.51 & -0.24 & -0.30 & 0.51 \\
0.96 & -0.01 & 0.59 & -0.59 & 0.09 & -0.29 & 1.07 & -0.13 & 0.35 & 0.53 & 0.70 & -1.12 & -0.33 & 0.35 \\
0.81 & 0.15 & 1.01 & -0.24 & 0.04 & 0.04 & 1.05 & -0.18 & -0.18 & 0.76 & 0.57 & -0.76 & -0.44 & 0.43 \\
0.74 & 0.18 & 0.45 & -0.72 & 0.00 & -0.14 & 1.29 & 0.07 & 0.03 & 0.41 & 0.33 & -1.09 & -0.27 & 0.38 \\
0.58 & 0.13 & 0.69 & -0.36 & -0.16 & 0.10 & 1.19 & -0.01 & -0.28 & 0.65 & 0.22 & -0.60 & -0.45 & 0.52
\end{bmatrix}$$

$$A_3 = \begin{bmatrix}
0.66 & -0.02 & 0.50 & -0.47 & -0.23 & -0.21 & 1.10 & -0.13 & 0.23 & 0.49 & 0.26 & -1.10 & -0.29 & 0.36 \\
0.49 & 0.17 & 0.68 & -0.52 & -0.34 & -0.01 & 1.07 & 0.05 & -0.20 & 0.67 & 0.22 & -0.77 & -0.51 & 0.45 \\
0.30 & 0.07 & 0.36 & -0.78 & -0.14 & -0.07 & 1.02 & -0.01 & 0.07 & 0.64 & 0.37 & -1.18 & -0.62 & 0.16 \\
0.43 & 0.15 & 0.35 & -0.58 & -0.33 & 0.09 & 0.99 & 0.20 & -0.36 & 0.85 & 0.48 & -0.87 & -0.84 & 0.45 \\
0.48 & -0.22 & 0.44 & -0.72 & -0.42 & -0.28 & 0.98 & -0.20 & -0.22 & 0.53 & 0.23 & -1.11 & -0.63 & 0.44 \\
0.54 & -0.14 & 0.34 & -0.68 & -0.65 & -0.09 & 0.94 & -0.18 & -0.52 & 0.56 & 0.34 & -0.89 & -0.71 & 0.59 \\
0.37 & -0.08 & 0.27 & -0.81 & -0.47 & -0.22 & 1.00 & -0.19 & -0.30 & 0.59 & 0.37 & -1.45 & -0.92 & 0.32 \\
0.49 & 0.06 & 0.22 & -0.79 & -0.65 & -0.10 & 0.89 & -0.29 & -0.52 & 0.65 & 0.57 & -1.16 & -1.02 & 0.54 \\
0.17 & -0.32 & 0.07 & -0.91 & -0.55 & 0.14 & 1.04 & 0.03 & -0.58 & 0.82 & 0.25 & -0.94 & -1.00 & 0.55 \\
0.36 & -027 & -0.02 & -0.97 & -0.56 & 0.15 & 1.02 & -0.05 & -0.62 & 0.75 & 0.36 & -0.99 & -1.11 & 0.74 \\
0.30 & -0.23 & 0.03 & -1.05 & -0.58 & 0.36 & 0.95 & 0.17 & -0.57 & 0.80 & 0.30 & -0.94 & -1.06 & 0.52 \\
0.42 & -0.16 & 0.02 & -1.16 & -0.61 & 0.30 & 1.08 & -0.12 & -0.67 & 0.88 & 0.42 & -0.98 & -1.17 & 0.65 \\
0.40 & -0.00 & -0.04 & -0.95 & -0.74 & 0.78 & 0.83 & 0.34 & -0.47 & 0.68 & 0.10 & -1.04 & -1.08 & 0.66 \\
0.46 & -0.10 & 0.13 & -1.33 & -0.75 & 0.62 & 1.05 & 0.08 & -0.67 & 0.70 & 0.18 & -1.13 & -1.12 & 0.69
\end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 2.24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.15 & 1.84 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.83 & -0.02 & 2.49 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.77 & 0.74 & 1.62 & 1.72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.74 & -0.02 & 2.20 & 0.50 & 2.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.66 & 0.50 & 1.26 & 0.71 & 1.15 & 1.64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.99 & -0.08 & 1.83 & 0.62 & 1.31 & 0.42 & 1.53 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.34 & 0.38 & 1.09 & 0.54 & 0.73 & 0.56 & 0.82 & 1.25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.75 & -0.17 & 1.67 & 0.54 & 1.41 & 0.34 & 0.64 & 0.20 & 1.50 & 0 & 0 & 0 & 0 & 0 \\ 0.44 & -0.00 & 1.01 & 0.67 & 0.58 & 0.71 & 0.28 & 0.49 & 0.95 & 1.22 & 0 & 0 & 0 & 0 \\ 0.90 & 0.07 & 1.27 & 0.58 & 1.26 & 0.24 & 0.86 & 0.23 & 0.75 & 0.16 & 1.28 & 0 & 0 & 0 \\ 0.59 & 0.27 & 0.97 & 0.96 & 0.54 & 0.42 & 0.56 & 0.30 & 0.31 & 0.37 & 1.11 & 1.08 & 0 & 0 \\ 0.88 & -0.20 & 1.26 & 0.55 & 0.88 & 0.12 & 1.00 & 0.25 & 1.03 & 0.08 & 0.31 & 0.05 & 1.29 & 0 \\ 0.61 & 0.02 & 1.02 & 0.74 & 0.48 & 0.42 & 0.70 & 0.25 & 0.54 & 0.43 & 0.15 & 0.43 & 0.79 & 0.99 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0.57 & -0.05 & 1.25 & 0.64 & 1.15 & 0.21 & 0.66 & 0.03 & 0.81 & 0.04 & 0.66 & 0.25 & 0.66 & 0.11 \\ 0.31 & 0.36 & 0.89 & 0.73 & 0.72 & 0.42 & 0.56 & 0.10 & 0.51 & 0.21 & 0.54 & 0.58 & 0.26 & 0.47 \\ 0.71 & -0.10 & 1.10 & 0.56 & 0.83 & 0.34 & 0.84 & 0.10 & 0.90 & 0.19 & 0.57 & 0.05 & 0.95 & 0.25 \\ 0.56 & 0.18 & 0.84 & 0.66 & 0.49 & 0.46 & 0.68 & 0.25 & 0.69 & 0.46 & 0.44 & 0.24 & 0.57 & 0.55 \\ 0.18 & 0.13 & 1.18 & 0.83 & 0.72 & 0.32 & 0.77 & 0.20 & 0.93 & 0.14 & 0.78 & 0.25 & 0.59 & 0.66 \\ 0.08 & 0.49 & 1.07 & 1.08 & 0.64 & 0.41 & 0.68 & 0.23 & 0.85 & 0.33 & 0.67 & 0.50 & 0.40 & 0.67 \\ 0.28 & 0.13 & 1.00 & 0.82 & 0.74 & 0.53 & 0.67 & 0.09 & 1.01 & 0.54 & 0.68 & 0.12 & 0.76 & 0.59 \\ 0.12 & 0.41 & 0.97 & 0.95 & 0.71 & 0.50 & 0.55 & 0.04 & 1.13 & 0.55 & 0.66 & 0.23 & 0.68 & 0.74 \\ 0.07 & 0.28 & 1.28 & 0.92 & 0.36 & 0.23 & 0.70 & 0.31 & 1.08 & 0.18 & 0.72 & 0.20 & 1.01 & 0.76 \\ -0.21 & 0.52 & 1.37 & 1.19 & 0.42 & 0.10 & 0.91 & 0.16 & 1.18 & 0.41 & 0.61 & 0.11 & 0.96 & 0.90 \\ 0.02 & 0.38 & 1.39 & 0.93 & 0.40 & 0.22 & 0.56 & 0.30 & 1.00 & 0.25 & 0.64 & 0.28 & 0.96 & 0.80 \\ -0.29 & 0.46 & 1.39 & 1.03 & 0.36 & 0.12 & 0.82 & 0.19 & 1.13 & 0.41 & 0.60 & 0.13 & 0.97 & 0.98 \\ 0.04 & 0.33 & 1.73 & 1.00 & 0.30 & 0.25 & 0.31 & 0.27 & 0.80 & 0.14 & 0.88 & 0.53 & 0.79 & 0.67 \\ -0.08 & 0.49 & 1.57 & 1.02 & 0.27 & 0.16 & 0.61 & -0.01 & 1.02 & 0.38 & 0.63 & 0.26 & 0.86 & 1.08 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 1.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.79 & 1.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.46 & 0.08 & 1.07 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0.65 & 0.77 & 0.75 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.57 & 0.47 & 0.23 & -0.05 & 1.09 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.44 & 0.91 & 0.10 & 0.50 & 0.73 & 0.78 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.72 & 0.30 & 0.55 & 0.26 & 0.35 & 0.12 & 1.17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.39 & 0.48 & 0.63 & 0.67 & 0.25 & 0.42 & 0.87 & 0.78 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.70 & 0.56 & 0.35 & 0.31 & 0.71 & 0.36 & 0.63 & 0.40 & 1.37 & 0 & 0 & 0 & 0 & 0 \\ 0.63 & 1.00 & 0.22 & 0.63 & 0.65 & 0.48 & 0.72 & 0.73 & 0.90 & 0.87 & 0 & 0 & 0 & 0 \\ 0.62 & 0.57 & 0.29 & 0.36 & 0.70 & 0.40 & 0.72 & 0.52 & 1.45 & 0.00 & 0.58 & 0 & 0 & 0 \\ 0.62 & 1.04 & 0.14 & 0.63 & 0.63 & 0.52 & 0.80 & 0.73 & 0.91 & 0.84 & 0.24 & 0.39 & 0 & 0 \\ 0.48 & 0.67 & 0.12 & 0.64 & 0.60 & 0.53 & 0.96 & 0.75 & 1.58 & -0.12 & 1.05 & 0.10 & 1.31 & 0 \\ 0.53 & 1.19 & 0.10 & 0.76 & 0.65 & 0.56 & 0.98 & 0.93 & 1.05 & 0.83 & 0.71 & 0.24 & 0.61 & 0.60 \end{bmatrix}$$

Cholesky decomposition of the parameter variance-covariance matrix from each of the estimated rate of decrease in central death rate equations, with the parameters in the following order: μ_κ, ϕ_{k_1}:

Boys age 0

$$\begin{bmatrix} 0.301443 & 0 \\ -0.029782 & 0.095315 \end{bmatrix}$$

Boys 1 to 4

$$\begin{bmatrix} 0.475674 & 0 \\ 0.040191 & 0.077393 \end{bmatrix}$$

Boys 5 to 9

$$\begin{bmatrix} 0.481556 & 0 \\ 0.025627 & 0.092059 \end{bmatrix}$$

Boys 10 to 14

$$\begin{bmatrix} 0.578565 & 0 \\ 0.015952 & 0.085717 \end{bmatrix}$$

Boys and Men 15 to 19

$$\begin{bmatrix} 0.846049 & 0 \\ 0.028506 & 0.081654 \end{bmatrix}$$

Men 20 to 24

$$\begin{bmatrix} 0.862519 & 0 \\ 0.037702 & 0.110139 \end{bmatrix}$$

Men 25 to 29

$$\begin{bmatrix} 1.046622 & 0 \\ 0.027024 & 0.062383 \end{bmatrix}$$

Men 30 to 34

$$\begin{bmatrix} 0.977386 & 0 \\ 0.028512 & 0.082742 \end{bmatrix}$$

Men 35 to 39

$$\begin{bmatrix} 0.675712 & 0 \\ 0.013943 & 0.146976 \end{bmatrix}$$

Men 40 to 44

$$\begin{bmatrix} 0.507667 & 0 \\ -0.050636 & 0.149500 \end{bmatrix}$$

Men 45 to 49

$$\begin{bmatrix} 0.360972 & 0 \\ -0.025556 & 0.108190 \end{bmatrix}$$

Men 50 to 54

$$\begin{bmatrix} 0.322319 & 0 \\ -0.015266 & 0.070982 \end{bmatrix}$$

Men 55 to 59

$$\begin{bmatrix} 0.325910 & 0 \\ 0.016391 & 0.065924 \end{bmatrix}$$

Men 60 to 64

$$\begin{bmatrix} 0.300593 & 0 \\ -0.008491 & 0.087176 \end{bmatrix}$$

Men 65 to 69

$$\begin{bmatrix} 0.273716 & 0 \\ 0.006429 & 0.106968 \end{bmatrix}$$

Men 70 to 74

$$\begin{bmatrix} 0.234159 & 0 \\ 0.021269 & 0.078799 \end{bmatrix}$$

Men 75 to 79

$$\begin{bmatrix} 0.246191 & 0 \\ 0.024471 & 0.080904 \end{bmatrix}$$

Men 80 to 84

$$\begin{bmatrix} 0.226873 & 0 \\ 0.020026 & 0.063280 \end{bmatrix}$$

Men 85 to 89

$$\begin{bmatrix} 0.268296 & 0 \\ -0.006886 & 0.062842 \end{bmatrix}$$

Men 90 to 94

$$\begin{bmatrix} 0.268629 & 0 \\ -0.004097 & 0.059072 \end{bmatrix}$$

Men 95 and over

$$\begin{bmatrix} 0.351433 & 0 \\ -0.016281 & 0.064372 \end{bmatrix}$$

Girls age 0

$$\begin{bmatrix} 0.286609 & 0 \\ -0.031943 & 0.083546 \end{bmatrix}$$

Girls 1 to 4

$$\begin{bmatrix} 0.512660 & 0 \\ 0.042046 & 0.071712 \end{bmatrix}$$

Girls 5 to 9

$$\begin{bmatrix} 0.555165 & 0 \\ 0.015179 & 0.065718 \end{bmatrix}$$

Girls 10 to 14

$$\begin{bmatrix} 0.694121 & 0 \\ 0.000339 & 0.081920 \end{bmatrix}$$

Girls and Women 15 to 19

$$\begin{bmatrix} 0.799380 & 0 \\ 0.018441 & 0.091412 \end{bmatrix}$$

Women 20 to 24

$$\begin{bmatrix} 0.989556 & 0 \\ 0.029893 & 0.068435 \end{bmatrix}$$

Women 25 to 29

$$\begin{bmatrix} 1.151190 & 0 \\ 0.028311 & 0.049099 \end{bmatrix}$$

Women 30 to 34

$$\begin{bmatrix} 0.912062 & 0 \\ 0.028093 & 0.073177 \end{bmatrix}$$

Women 35 to 39

$$\begin{bmatrix} 0.599478 & 0 \\ 0.021685 & 0.127704 \end{bmatrix}$$

Women 40 to 44

$$\begin{bmatrix} 0.422401 & 0 \\ -0.006536 & 0.148542 \end{bmatrix}$$

Women 45 to 49

$$\begin{bmatrix} 0.321234 & 0 \\ -0.014699 & 0.161921 \end{bmatrix}$$

Women 50 to 54

$$\begin{bmatrix} 0.246731 & 0 \\ 0.017169 & 0.096181 \end{bmatrix}$$

Women 55 to 59

$$\begin{bmatrix} 0.269373 & 0 \\ -0.013094 & 0.085901 \end{bmatrix}$$

Women 60 to 64

$$\begin{bmatrix} 0.239761 & 0 \\ -0.014617 & 0.106417 \end{bmatrix}$$

Women 65 to 69

$$\begin{bmatrix} 0.226724 & 0 \\ 0.018589 & 0.101835 \end{bmatrix}$$

Women 70 to 74

$$\begin{bmatrix} 0.207535 & 0 \\ 0.003028 & 0.085607 \end{bmatrix}$$

Women 75 to 79

$$\begin{bmatrix} 0.235224 & 0 \\ 0.027749 & 0.074606 \end{bmatrix}$$

Women 80 to 84

$$\begin{bmatrix} 0.236155 & 0 \\ 0.016700 & 0.064967 \end{bmatrix}$$

Women 85 to 89

$$\begin{bmatrix} 0.313665 & 0 \\ -0.005681 & 0.067138 \end{bmatrix}$$

Women 90 to 94

$$\begin{bmatrix} 0.310660 & 0 \\ -0.006154 & 0.068316 \end{bmatrix}$$

Women 95 and over

$$\begin{bmatrix} 0.346683 & 0 \\ -0.001743 & 0.064907 \end{bmatrix}$$

LPR New Arrival Immigration

$L_t = \mu + \phi_1(L_{t-1} - \mu) + \varepsilon_t + \theta_1\varepsilon_{t-1},$

Note: An upper bound for the LPR new arrival immigration was set at 30 million.

AR parameters	MA parameters	Regression Mean Over Historical Period	Standard Error
ϕ_1 = 0.716520	θ_1 = 0. 594647	μ = 535,440.7	σ_ε = 59,777.54

Cholesky decomposition of the parameter variance-covariance matrix from the estimated legal immigration equation, with the parameters in the following order: μ, ϕ_1, θ_1:

$$\begin{bmatrix} 63622.76 & 0 & 0 \\ 0.002365 & 0.116440 & 0 \\ -0.065367 & -0.063506 & 0.152604 \end{bmatrix}$$

Adjustments of Status (Transfers)

$S_t = \mu + \phi_1(S_{t-1} - \mu) + \varepsilon_t,$

In the equation above, S_t represents the ratio of the number of transfers to the number of other-than-LPR immigrants and is expressed as a log-odds ratio.

AR parameters	Regression Mean Over Historical Period	Standard Error
ϕ_1 = 0.702792	μ = -3.390283	σ_ε = 0.189973

Cholesky decomposition of the parameter variance-covariance matrix from the estimated transfers equation, with the parameters in the following order: μ, ϕ_1:

$$\begin{bmatrix} 0.085554 & 0 \\ -0.023485 & 0.061967 \end{bmatrix}$$

Legal Emigration

$E_t = \mu + \phi_1(E_{t-1} - \mu) + \phi_2(E_{t-2} - \mu) + \phi_3(E_{t-3} - \mu) + \varepsilon_t,$

In the equation above, E_t represents the ratio of the number of legal emigrants to the total number of LPR immigrants plus citizens and is expressed as a log-odds ratio.

AR parameters	Regression Mean Over Historical Period	Standard Error
ϕ_1 = 1.269066	μ = -7.375162	σ_ε = 0.115638
ϕ_2= -0.633007		
ϕ_3 = 0.233052		

Cholesky decomposition of the parameter variance-covariance matrix from the estimated legal emigration equation, with the parameters in the following order: μ, ϕ_1, ϕ_2, ϕ_3:

$$\begin{bmatrix} 0.101544 & 0 & 0 & 0 \\ -0.036127 & 0.133759 & 0 & 0 \\ 0.023605 & -0.164998 & 0.072390 & 0 \\ 0.017126 & 0.044731 & -0.065557 & 0.026498 \end{bmatrix}$$

Other-than-LPR Immigration

$$O_t = \mu + \phi_1(O_{t-1} - \mu) + \varepsilon_t,$$

Note: A lower bound for the number of other-than-LPR immigrants arriving was set at 100,000 and an upper bound was set at 15 million.

AR parameters	Regression Mean Over Historical Period	Standard Error
$\phi_1 = 0.656336$	$\mu = 978{,}691.8$	$\sigma_\varepsilon = 233{,}249.9$

Cholesky decomposition of the parameter variance-covariance matrix from the estimated other-than-LPR immigration equation, with the parameters in the following order: μ, ϕ_1:

$$\begin{bmatrix} 161512.8 & 0 \\ -0.033522 & 0.130315 \end{bmatrix}$$

Economic VAR Framework

Means from the VAR

$\mu_U = -2.753272$
$\mu_I = -2.804968$
$\mu_R = 0.024333$

Unemployment Rate (expressed as log-odds ratio)

$$U_t = \mu_U + a_{111}(U_{t-1} - \mu_U) + a_{112}(U_{t-2} - \mu_U) + a_{121}(I_{t-1} - \mu_I) + a_{122}(I_{t-2} - \mu_I) + a_{131}(R_{t-1} - \mu_R) + a_{132}(R_{t-2} - \mu_R) + \varepsilon_{1t}$$

Unemployment Mean and AR Parameters

$a_{111} = 1.103985$	$a_{112} = -0.439009$
$a_{121} = 0.086839$	$a_{122} = 0.076847$
$a_{131} = -0.908053$	$a_{132} = 0.110108$

Inflation Rate (transformed into logs with lower bound of -3.0 percent)

$$I_t = \mu_I + a_{211}(U_{t-1} - \mu_U) + a_{212}(U_{t-2} - \mu_U) + a_{221}(I_{t-1} - \mu_I) + a_{222}(I_{t-2} - \mu_I) + a_{231}(R_{t-1} - \mu_R) + a_{232}(R_{t-2} - \mu_R) + \varepsilon_{2t}$$

Note: In cases where the nominal interest rate is less than 0 when combining the inflation rate and real interest rate, the real interest rate is set such that the nominal interest rate is 0.

<u>Constant and AR Parameters</u>

a_{211} = -0.205316	a_{212} = 0.156342
a_{221} = 0.303731	a_{222} = 0.525542
a_{231} = -6.815745	a_{232} = 6.568878

Real Interest Rate

$$R_t = \mu_R + a_{311}(U_{t-1} - \mu_U) + a_{312}(U_{t-2} - \mu_U) + a_{321}(I_{t-1} - \mu_I) + a_{322}(I_{t-2} - \mu_I) + a_{331}(R_{t-1} - \mu_R) + a_{332}(R_{t-2} - \mu_R) + \varepsilon_{3t}$$

Note: Bounds of plus and minus 40 percent were put on the inflation rate and real interest rate. For the real interest rate, the plus/minus 40-percent bound is imposed before the bound requiring a nominal interest rate of at least 0 percent.

<u>Constant and AR Parameters</u>

a_{311} = 0.005701	a_{312} = -0.005288
a_{321} = 0.041293	a_{322} = -0.024433
a_{331} = 1.281961	a_{332} = -0.427983

VAR Matrix Notation

$$\begin{bmatrix} U_t \\ I_t \\ R_t \end{bmatrix} = \begin{bmatrix} \mu_U \\ \mu_I \\ \mu_R \end{bmatrix} + A_1 \begin{bmatrix} U_{t-1} - \mu_U \\ I_{t-1} - \mu_I \\ R_{t-1} - \mu_R \end{bmatrix} + A_2 \begin{bmatrix} U_{t-2} - \mu_U \\ I_{t-2} - \mu_I \\ R_{t-2} - \mu_R \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} a_{112} & a_{122} & a_{132} \\ a_{212} & a_{222} & a_{232} \\ a_{312} & a_{322} & a_{332} \end{bmatrix}$$

Cholesky decomposition of the residual variance-covariance matrix:

$$\begin{bmatrix} 0.152470 & 0 & 0 \\ -0.061977 & 0.222491 & 0 \\ 0.002770 & -0.010109 & 0.008994 \end{bmatrix}$$

Cholesky decomposition of the parameter variance-covariance matrix from the vector autoregression, with the parameters in the following order:

1. Regression mean from the unemployment rate equation
2. Unemployment rate lagged one period from the unemployment rate equation
3. Unemployment rate lagged two periods from the unemployment rate equation
4. CPI lagged one period from the unemployment rate equation
5. CPI lagged two periods from the unemployment rate equation
6. Real interest rate lagged one period from the unemployment rate equation
7. Real interest rate lagged two periods from the unemployment rate equation
8. Regression mean from the CPI equation
9. Unemployment rate lagged one period from the CPI equation
10. Unemployment rate lagged two periods from the CPI equation
11. CPI lagged one period from the CPI equation
12. CPI lagged two periods from the CPI equation
13. Real interest rate lagged one period from the CPI equation
14. Real interest rate lagged two periods from the CPI equation
15. Regression mean from the real interest rate equation
16. Unemployment rate lagged one period from the real interest rate equation
17. Unemployment rate lagged two periods from the real interest rate equation
18. CPI lagged one period from the real interest rate equation
19. CPI lagged two periods from the real interest rate equation
20. Real interest rate lagged one period from the real interest rate equation
21. Real interest rate lagged two periods from the real interest rate equation

$$\mathbf{V} = \mathbf{L}\mathbf{L}' \qquad \mathbf{L} = \begin{bmatrix} \boldsymbol{C_1} & \boldsymbol{D_1} \\ \boldsymbol{C_2} & \boldsymbol{D_2} \end{bmatrix}$$

(**L** is a 21 x 21 matrix)

$$\boldsymbol{C_1} = \begin{bmatrix}
0.067 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.042 & 0.189 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.048 & -0.123 & 0.071 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.008 & 0.024 & 0.033 & 0.058 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.072 & -0.040 & 0.069 & -0.022 & 0.106 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.037 & -0.045 & -0.026 & -0.058 & 0.004 & 0.040 & 0 & 0 & 0 & 0 & 0 \\
-0.121 & -0.589 & 0.096 & 0.826 & -0.205 & -0.301 & 1.036 & 0 & 0 & 0 & 0 \\
0.001 & -0.003 & 0.006 & 0.000 & 0.005 & 0.002 & 0.003 & 0.005 & 0 & 0 & 0 \\
-0.063 & 0.562 & -0.029 & -0.776 & 0.358 & 0.373 & -1.095 & -0.125 & 0.502 & 0 & 0 \\
0.027 & -0.012 & 0.086 & -0.032 & -0.120 & -0.063 & 0.066 & 0.028 & -0.015 & 0.203 & 0 \\
0.069 & 0.035 & -0.118 & 0.041 & 0.142 & 0.050 & -0.042 & 0.010 & 0.021 & -0.185 & 0.068
\end{bmatrix}$$

$$\boldsymbol{C_2} = \begin{bmatrix} -0.052 & -0.043 & 0.037 & 0.029 & -0.016 & -0.017 & 0.013 & 0.003 & -0.021 & 0.040 & 0.003 \\ 0.011 & -0.022 & -0.046 & 0.002 & 0.061 & 0.018 & -0.041 & -0.041 & 0.010 & -0.091 & -0.051 \\ -0.526 & 1.044 & -0.291 & 0.118 & -0.753 & -0.084 & 0.300 & 0.832 & 0.081 & -0.021 & 0.164 \\ 0.003 & -0.896 & 0.120 & -0.186 & 0.215 & -0.209 & -0.355 & -1.003 & -0.536 & 0.413 & -0.715 \\ 0.002 & -0.001 & -0.005 & 0.001 & 0.009 & -0.000 & -0.005 & -0.003 & -0.000 & -0.008 & -0.002 \\ -0.006 & -0.000 & 0.006 & -0.000 & -0.010 & 0.002 & 0.005 & 0.002 & -0.000 & 0.007 & -0.000 \\ 0.001 & -0.002 & 0.000 & 0.002 & 0.002 & 0.001 & 0.000 & 0.000 & -0.001 & -0.003 & -0.001 \\ -0.000 & 0.004 & 0.001 & -0.001 & -0.006 & 0.000 & 0.002 & 0.003 & 0.002 & 0.006 & 0.003 \\ 0.032 & -0.066 & 0.033 & 0.033 & 0.075 & 0.004 & -0.007 & -0.022 & -0.055 & -0.022 & -0.024 \\ -0.007 & 0.061 & -0.030 & -0.038 & -0.053 & -0.003 & -0.001 & 0.018 & 0.075 & 0.009 & 0.050 \end{bmatrix}$$

$$\boldsymbol{D_1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{D_2} = \begin{bmatrix} 0.123 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.123 & 0.049 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.245 & -0.595 & 0.823 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.930 & 0.590 & -0.609 & 0.553 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.000 & 0.002 & -0.001 & 0.000 & 0.003 & 0 & 0 & 0 & 0 & 0 \\ -0.001 & -0.002 & 0.001 & -0.000 & -0.004 & 0.002 & 0 & 0 & 0 & 0 \\ -0.004 & 0.001 & 0.001 & 0.001 & 0.001 & -0.000 & 0.003 & 0 & 0 & 0 \\ 0.004 & -0.003 & 0.000 & -0.002 & -0.001 & 0.001 & -0.001 & 0.002 & 0 & 0 \\ -0.035 & 0.063 & -0.028 & 0.034 & -0.009 & -0.005 & 0.031 & -0.021 & 0.053 & 0 \\ 0.035 & -0.053 & 0.019 & -0.071 & 0.016 & -0.001 & -0.043 & 0.021 & -0.045 & 0.024 \end{bmatrix}$$

Real Average Wage (percent change)

$$W_t = \beta_0 + \beta_1 u_t + \beta_2 u_{t-1} + \varepsilon_t$$

Note: The annual percent change in the real average wage is estimated as a function of the current and lagged unemployment rates (expressed as log-odds ratios) deviating from the historical mean.

Estimated Parameters	Standard Error
$\beta_0 = 0.998129$	$\sigma_\varepsilon = 1.569990$
$\beta_1 = -1.530037$	
$\beta_2 = -0.393138$	

Cholesky decomposition of the parameter variance-covariance matrix from the estimated real average wage equation, with the parameters in the following order: mean of the regression (β_0), unemployment rate, lagged unemployment rate:

$$\begin{bmatrix} 0.210740 & 0 & 0 \\ 0.729548 & 2.566081 & 0 \\ -0.951809 & -2.412212 & 0.514165 \end{bmatrix}$$

Disability Incidence – Male (expressed as log-odds ratio)

$$DIM_t = \mu + \phi_1(DIM_{t-1} - \mu) + \phi_2(DIM_{t-2} - \mu) + \varepsilon_t,$$

AR parameters	Regression Mean Over Historical Period	Standard Error
ϕ_1 = 1.287210	μ = -3.065424	σ_ε = 0.081089
ϕ_2 = -0.344726		

Cholesky decomposition of the parameter variance-covariance matrix from the estimated male disability incidence equation, with the parameters in the following order: μ, ϕ_1, ϕ_2:

$$\begin{bmatrix} 0.360431 & 0 & 0 \\ 0.003622 & 0.210198 & 0 \\ -0.065427 & -0.203654 & 0.040104 \end{bmatrix}$$

Disability Incidence – Female (expressed as log-odds ratio)

$$DIF_t = \mu + \phi_1(DIF_{t-1} - \mu) + \phi_2(DIF_{t-2} - \mu) + \varepsilon_t,$$

AR parameters	Regression Mean Over Historical Period	Standard Error
ϕ_1 = 1.345328	μ = -3.113388	σ_ε = 0.081417
ϕ_2 = -0.439248		

Cholesky decomposition of the parameter variance-covariance matrix from the estimated female disability incidence equation, with the parameters in the following order: μ, ϕ_1, ϕ_2:

$$\begin{bmatrix} 0.140141 & 0 & 0 \\ 0.000497 & 0.202243 & 0 \\ -0.030797 & -0.195458 & 0.045644 \end{bmatrix}$$

Cholesky decomposition of the residual variance-covariance matrix:

$$\begin{bmatrix} 0.081089 & 0 \\ 0.077243 & 0.025735 \end{bmatrix}$$

Disability Recovery – Male (expressed as log-odds ratio)

$DRM_t = \mu + \phi_1(DRM_{t-1} - \mu) + \varepsilon_t,$

AR parameters	Regression Mean Over Historical Period	Standard Error
$\phi_1 = 0.619281$	$\mu = -2.167094$	$\sigma_\varepsilon = 0.258440$

Cholesky decomposition of the parameter variance-covariance matrix from the estimated male disability recovery equation, with the parameters in the following order: μ, ϕ_1:

$$\begin{bmatrix} 0.126645 & 0 \\ 0.100781 & 0.216991 \end{bmatrix}$$

Disability Recovery – Female (expressed as log-odds ratio)

$DRF_t = \mu + \phi_1(DRF_{t-1} - \mu) + \varepsilon_t,$

AR parameters	Regression Mean Over Historical Period	Standard Error
$\phi_1 = 0.711241$	$\mu = -2.216565$	$\sigma_\varepsilon = 0.238036$

Cholesky decomposition of the parameter variance-covariance matrix from the estimated female disability recovery equation, with the parameters in the following order: μ, ϕ_1:

$$\begin{bmatrix} 0.151912 & 0 \\ 0.080653 & 0.171401 \end{bmatrix}$$

Cholesky decomposition of the residual variance-covariance matrix:

$$\begin{bmatrix} 0.259058 & 0 \\ 0.233211 & 0.050449 \end{bmatrix}$$

2. Assumed Annual Projected Values for Each Variable from the 2023 Trustees Report

t	F_t^{TR}	$M_{1,t}^{TR}$	$M_{2,t}^{TR}$	$M_{3,t}^{TR}$	$M_{4,t}^{TR}$	$M_{5,t}^{TR}$	$M_{6,t}^{TR}$	$M_{7,t}^{TR}$	$M_{8,t}^{TR}$
2023	1.7038	1.4913	1.5398	5.2051	5.0485	4.4623	4.6125	4.8200	4.3476
2024	1.7225	1.5218	1.5623	2.4098	2.2891	2.2322	1.9338	2.0970	1.6997
2025	1.7421	1.5464	1.5803	1.4205	1.3311	1.4456	1.0480	1.1721	0.8428
2026	1.7631	1.5661	1.5947	1.4097	1.3463	1.4409	1.1238	1.2154	0.9438
2027	1.7853	1.5818	1.6062	1.4008	1.3581	1.4370	1.1843	1.2498	1.0245
2028	1.8085	1.5944	1.6153	1.3934	1.3673	1.4336	1.2324	1.2772	1.0889
2029	1.8325	1.6044	1.6225	1.3873	1.3744	1.4307	1.2707	1.2988	1.1402
2030	1.8572	1.6124	1.6283	1.3820	1.3798	1.4281	1.3010	1.3158	1.1810
2031	1.8687	1.6186	1.6328	1.3775	1.3838	1.4257	1.3249	1.3291	1.2133
2032	1.8811	1.6236	1.6363	1.3736	1.3867	1.4236	1.3438	1.3394	1.2390
2033	1.8940	1.6274	1.6390	1.3702	1.3887	1.4216	1.3586	1.3474	1.2592
2034	1.9071	1.6304	1.6411	1.3671	1.3900	1.4198	1.3701	1.3535	1.2751
2035	1.9202	1.6327	1.6427	1.3644	1.3908	1.4180	1.3790	1.3581	1.2875
2036	1.9330	1.6344	1.6439	1.3619	1.3911	1.4164	1.3858	1.3614	1.2972
2037	1.9450	1.6357	1.6447	1.3596	1.3910	1.4148	1.3909	1.3638	1.3046
2038	1.9559	1.6366	1.6453	1.3575	1.3907	1.4133	1.3947	1.3654	1.3102
2039	1.9656	1.6373	1.6456	1.3555	1.3901	1.4118	1.3974	1.3664	1.3144
2040	1.9738	1.6377	1.6458	1.3536	1.3894	1.4104	1.3993	1.3669	1.3175
2041	1.9806	1.6379	1.6459	1.3517	1.3885	1.4090	1.4005	1.3669	1.3197
2042	1.9861	1.6379	1.6458	1.3500	1.3875	1.4076	1.4011	1.3667	1.3211
2043	1.9903	1.6378	1.6457	1.3483	1.3864	1.4062	1.4013	1.3662	1.3219
2044	1.9934	1.6377	1.6455	1.3467	1.3853	1.4049	1.4012	1.3655	1.3223
2045	1.9957	1.6374	1.6452	1.3451	1.3841	1.4036	1.4008	1.3647	1.3223
2046	1.9973	1.6371	1.6449	1.3435	1.3828	1.4022	1.4001	1.3637	1.3220
2047	1.9983	1.6376	1.6451	1.3415	1.3820	1.4007	1.4021	1.3642	1.3253
2048	1.9990	1.6370	1.6446	1.3401	1.3806	1.3995	1.4005	1.3627	1.3238
2049	1.9994	1.6365	1.6441	1.3387	1.3792	1.3982	1.3990	1.3612	1.3223
2050	1.9997	1.6359	1.6436	1.3372	1.3777	1.3970	1.3974	1.3598	1.3208
2051	1.9998	1.6353	1.6430	1.3358	1.3763	1.3957	1.3959	1.3583	1.3193
2052	1.9999	1.6348	1.6425	1.3344	1.3748	1.3944	1.3943	1.3568	1.3178
2053	1.9999	1.6342	1.6420	1.3329	1.3734	1.3932	1.3927	1.3553	1.3163
2054	2.0000	1.6336	1.6415	1.3315	1.3719	1.3919	1.3912	1.3538	1.3148
2055	2.0000	1.6330	1.6410	1.3301	1.3705	1.3907	1.3896	1.3523	1.3133
2056	2.0000	1.6324	1.6404	1.3286	1.3691	1.3894	1.3880	1.3508	1.3118
2057	2.0000	1.6319	1.6399	1.3272	1.3676	1.3881	1.3865	1.3493	1.3103
2058	2.0000	1.6313	1.6394	1.3258	1.3662	1.3869	1.3849	1.3478	1.3088
2059	2.0000	1.6307	1.6388	1.3244	1.3647	1.3856	1.3833	1.3463	1.3073
2060	2.0000	1.6301	1.6383	1.3229	1.3633	1.3844	1.3817	1.3448	1.3059
2061	2.0000	1.6295	1.6378	1.3215	1.3618	1.3831	1.3801	1.3433	1.3044
2062	2.0000	1.6289	1.6372	1.3201	1.3604	1.3819	1.3786	1.3418	1.3029
2063	2.0000	1.6283	1.6367	1.3187	1.3589	1.3806	1.3770	1.3403	1.3014
2064	2.0000	1.6277	1.6361	1.3173	1.3575	1.3793	1.3754	1.3387	1.2999
2065	2.0000	1.6271	1.6356	1.3158	1.3561	1.3781	1.3738	1.3372	1.2984
2066	2.0000	1.6265	1.6350	1.3144	1.3546	1.3768	1.3722	1.3357	1.2969
2067	2.0000	1.6259	1.6345	1.3130	1.3532	1.3756	1.3706	1.3342	1.2954
2068	2.0000	1.6253	1.6339	1.3116	1.3517	1.3743	1.3690	1.3327	1.2939
2069	2.0000	1.6247	1.6334	1.3102	1.3503	1.3731	1.3674	1.3312	1.2924
2070	2.0000	1.6241	1.6328	1.3088	1.3488	1.3718	1.3658	1.3297	1.2909
2071	2.0000	1.6235	1.6322	1.3074	1.3474	1.3706	1.3642	1.3282	1.2894
2072	2.0000	1.6229	1.6317	1.3060	1.3459	1.3693	1.3626	1.3267	1.2879
2073	2.0000	1.6223	1.6311	1.3045	1.3445	1.3681	1.3609	1.3252	1.2864
2074	2.0000	1.6216	1.6305	1.3031	1.3431	1.3668	1.3593	1.3237	1.2849
2075	2.0000	1.6210	1.6300	1.3017	1.3416	1.3656	1.3577	1.3222	1.2835
2076	2.0000	1.6204	1.6294	1.3003	1.3402	1.3643	1.3561	1.3207	1.2820
2077	2.0000	1.6198	1.6288	1.2989	1.3387	1.3631	1.3545	1.3191	1.2805
2078	2.0000	1.6191	1.6282	1.2975	1.3373	1.3619	1.3528	1.3176	1.2790
2079	2.0000	1.6185	1.6276	1.2961	1.3359	1.3606	1.3512	1.3161	1.2775
2080	2.0000	1.6179	1.6271	1.2947	1.3344	1.3594	1.3496	1.3146	1.2760
2081	2.0000	1.6172	1.6265	1.2934	1.3330	1.3581	1.3480	1.3131	1.2745
2082	2.0000	1.6166	1.6259	1.2920	1.3315	1.3569	1.3463	1.3116	1.2730
2083	2.0000	1.6159	1.6253	1.2906	1.3301	1.3556	1.3447	1.3101	1.2716
2084	2.0000	1.6153	1.6247	1.2892	1.3287	1.3544	1.3430	1.3086	1.2701
2085	2.0000	1.6146	1.6241	1.2878	1.3272	1.3532	1.3414	1.3070	1.2686
2086	2.0000	1.6140	1.6235	1.2864	1.3258	1.3519	1.3398	1.3055	1.2671
2087	2.0000	1.6133	1.6229	1.2850	1.3243	1.3507	1.3381	1.3040	1.2656

t	F_t^{TR}	$M_{1,t}^{TR}$	$M_{2,t}^{TR}$	$M_{3,t}^{TR}$	$M_{4,t}^{TR}$	$M_{5,t}^{TR}$	$M_{6,t}^{TR}$	$M_{7,t}^{TR}$	$M_{8,t}^{TR}$
2088	2.0000	1.6127	1.6223	1.2836	1.3229	1.3495	1.3365	1.3025	1.2641
2089	2.0000	1.6120	1.6217	1.2823	1.3215	1.3482	1.3348	1.3010	1.2627
2090	2.0000	1.6113	1.6210	1.2809	1.3200	1.3470	1.3332	1.2995	1.2612
2091	2.0000	1.6107	1.6204	1.2795	1.3186	1.3457	1.3315	1.2980	1.2597
2092	2.0000	1.6100	1.6198	1.2781	1.3172	1.3445	1.3298	1.2964	1.2582
2093	2.0000	1.6093	1.6192	1.2768	1.3157	1.3433	1.3282	1.2949	1.2567
2094	2.0000	1.6087	1.6186	1.2754	1.3143	1.3420	1.3265	1.2934	1.2553
2095	2.0000	1.6080	1.6179	1.2740	1.3129	1.3408	1.3249	1.2919	1.2538
2096	2.0000	1.6073	1.6173	1.2727	1.3114	1.3396	1.3232	1.2904	1.2523
2097	2.0000	1.6066	1.6167	1.2713	1.3100	1.3384	1.3215	1.2889	1.2508

t	$M_{9,t}^{TR}$	$M_{10,t}^{TR}$	$M_{11,t}^{TR}$	$M_{12,t}^{TR}$	$M_{13,t}^{TR}$	$M_{14,t}^{TR}$	$M_{15,t}^{TR}$	$M_{16,t}^{TR}$	$M_{17,t}^{TR}$
2023	4.3556	7.6346	7.5666	7.2803	7.4677	6.9534	6.9462	6.7615	6.6349
2024	1.6985	2.4645	2.3970	2.1522	2.3040	1.8805	1.8630	1.7275	1.6025
2025	0.8339	1.5759	1.5120	1.3121	1.4289	1.0979	1.0737	0.9838	0.8673
2026	0.9286	0.6384	0.5773	0.4143	0.5023	0.2466	0.2167	0.1639	0.0542
2027	1.0042	0.6759	0.6178	0.4861	0.5502	0.3575	0.3236	0.3008	0.1984
2028	1.0646	0.7058	0.6501	0.5437	0.5885	0.4464	0.4094	0.4105	0.3142
2029	1.1126	0.7296	0.6758	0.5899	0.6192	0.5177	0.4782	0.4983	0.4071
2030	1.1509	0.7485	0.6963	0.6269	0.6437	0.5749	0.5333	0.5686	0.4817
2031	1.1812	0.7635	0.7127	0.6565	0.6634	0.6205	0.5775	0.6248	0.5415
2032	1.2052	0.7753	0.7257	0.6801	0.6790	0.6571	0.6128	0.6696	0.5894
2033	1.2242	0.7847	0.7360	0.6990	0.6915	0.6862	0.6411	0.7054	0.6277
2034	1.2391	0.7920	0.7441	0.7141	0.7015	0.7095	0.6637	0.7340	0.6583
2035	1.2508	0.7977	0.7505	0.7261	0.7094	0.7280	0.6817	0.7567	0.6828
2036	1.2599	0.8022	0.7556	0.7356	0.7157	0.7427	0.6961	0.7747	0.7023
2037	1.2669	0.8056	0.7596	0.7431	0.7207	0.7544	0.7076	0.7890	0.7179
2038	1.2722	0.8082	0.7626	0.7491	0.7246	0.7637	0.7167	0.8003	0.7302
2039	1.2763	0.8101	0.7650	0.7538	0.7278	0.7710	0.7239	0.8092	0.7401
2040	1.2792	0.8115	0.7668	0.7574	0.7302	0.7767	0.7297	0.8162	0.7478
2041	1.2813	0.8125	0.7681	0.7603	0.7321	0.7812	0.7342	0.8216	0.7540
2042	1.2828	0.8132	0.7691	0.7625	0.7336	0.7847	0.7378	0.8258	0.7588
2043	1.2837	0.8136	0.7698	0.7642	0.7347	0.7874	0.7406	0.8290	0.7626
2044	1.2841	0.8138	0.7703	0.7655	0.7356	0.7894	0.7428	0.8315	0.7656
2045	1.2842	0.8138	0.7706	0.7664	0.7362	0.7909	0.7444	0.8333	0.7678
2046	1.2841	0.8136	0.7707	0.7671	0.7367	0.7921	0.7457	0.8346	0.7696
2047	1.2873	0.8152	0.7727	0.7710	0.7393	0.7982	0.7519	0.8420	0.7779
2048	1.2860	0.8145	0.7722	0.7706	0.7390	0.7977	0.7516	0.8413	0.7775
2049	1.2847	0.8138	0.7718	0.7702	0.7388	0.7971	0.7513	0.8406	0.7770
2050	1.2835	0.8132	0.7713	0.7698	0.7386	0.7966	0.7510	0.8399	0.7766
2051	1.2822	0.8125	0.7709	0.7694	0.7383	0.7961	0.7507	0.8392	0.7762
2052	1.2809	0.8118	0.7705	0.7690	0.7381	0.7956	0.7504	0.8384	0.7758
2053	1.2797	0.8112	0.7700	0.7686	0.7379	0.7950	0.7501	0.8377	0.7753
2054	1.2784	0.8105	0.7696	0.7682	0.7376	0.7945	0.7498	0.8370	0.7749
2055	1.2771	0.8099	0.7692	0.7678	0.7374	0.7940	0.7495	0.8363	0.7745
2056	1.2759	0.8092	0.7688	0.7674	0.7372	0.7935	0.7492	0.8356	0.7741
2057	1.2746	0.8086	0.7683	0.7671	0.7369	0.7930	0.7489	0.8349	0.7737
2058	1.2734	0.8080	0.7679	0.7667	0.7367	0.7925	0.7486	0.8342	0.7733
2059	1.2721	0.8073	0.7675	0.7663	0.7365	0.7920	0.7483	0.8335	0.7729
2060	1.2709	0.8067	0.7671	0.7659	0.7362	0.7915	0.7481	0.8328	0.7725
2061	1.2696	0.8061	0.7667	0.7655	0.7360	0.7910	0.7478	0.8321	0.7721
2062	1.2684	0.8054	0.7662	0.7652	0.7358	0.7905	0.7475	0.8315	0.7717
2063	1.2671	0.8048	0.7658	0.7648	0.7356	0.7900	0.7472	0.8308	0.7713
2064	1.2659	0.8042	0.7654	0.7644	0.7354	0.7895	0.7470	0.8301	0.7709
2065	1.2647	0.8036	0.7650	0.7641	0.7351	0.7890	0.7467	0.8294	0.7705
2066	1.2634	0.8030	0.7646	0.7637	0.7349	0.7885	0.7464	0.8287	0.7701
2067	1.2622	0.8024	0.7642	0.7633	0.7347	0.7880	0.7461	0.8281	0.7697
2068	1.2610	0.8018	0.7638	0.7630	0.7345	0.7876	0.7459	0.8274	0.7693
2069	1.2597	0.8011	0.7634	0.7626	0.7343	0.7871	0.7456	0.8267	0.7689
2070	1.2585	0.8005	0.7630	0.7622	0.7341	0.7866	0.7453	0.8261	0.7685
2071	1.2573	0.7999	0.7626	0.7619	0.7338	0.7861	0.7451	0.8254	0.7682
2072	1.2561	0.7994	0.7623	0.7615	0.7336	0.7857	0.7448	0.8248	0.7678
2073	1.2548	0.7988	0.7619	0.7612	0.7334	0.7852	0.7445	0.8241	0.7674
2074	1.2536	0.7982	0.7615	0.7608	0.7332	0.7847	0.7443	0.8235	0.7670
2075	1.2524	0.7976	0.7611	0.7605	0.7330	0.7843	0.7440	0.8228	0.7666
2076	1.2512	0.7970	0.7607	0.7601	0.7328	0.7838	0.7438	0.8222	0.7663
2077	1.2500	0.7964	0.7604	0.7598	0.7326	0.7833	0.7435	0.8215	0.7659
2078	1.2488	0.7958	0.7600	0.7594	0.7324	0.7829	0.7433	0.8209	0.7655
2079	1.2476	0.7953	0.7596	0.7591	0.7322	0.7824	0.7430	0.8203	0.7652
2080	1.2463	0.7947	0.7592	0.7587	0.7320	0.7820	0.7427	0.8196	0.7648
2081	1.2451	0.7941	0.7589	0.7584	0.7318	0.7815	0.7425	0.8190	0.7644
2082	1.2439	0.7936	0.7585	0.7581	0.7316	0.7811	0.7422	0.8184	0.7641
2083	1.2427	0.7930	0.7581	0.7577	0.7314	0.7806	0.7420	0.8178	0.7637
2084	1.2415	0.7925	0.7578	0.7574	0.7312	0.7802	0.7418	0.8171	0.7633
2085	1.2403	0.7919	0.7574	0.7571	0.7310	0.7797	0.7415	0.8165	0.7630
2086	1.2392	0.7913	0.7571	0.7567	0.7308	0.7793	0.7413	0.8159	0.7626
2087	1.2380	0.7908	0.7567	0.7564	0.7306	0.7789	0.7410	0.8153	0.7623
2088	1.2368	0.7902	0.7563	0.7561	0.7304	0.7784	0.7408	0.8147	0.7619
2089	1.2356	0.7897	0.7560	0.7558	0.7303	0.7780	0.7405	0.8141	0.7616

t	$M_{9,t}^{TR}$	$M_{10,t}^{TR}$	$M_{11,t}^{TR}$	$M_{12,t}^{TR}$	$M_{13,t}^{TR}$	$M_{14,t}^{TR}$	$M_{15,t}^{TR}$	$M_{16,t}^{TR}$	$M_{17,t}^{TR}$
2090	1.2344	0.7892	0.7556	0.7554	0.7301	0.7776	0.7403	0.8135	0.7612
2091	1.2332	0.7886	0.7553	0.7551	0.7299	0.7771	0.7401	0.8129	0.7609
2092	1.2320	0.7881	0.7549	0.7548	0.7297	0.7767	0.7398	0.8123	0.7605
2093	1.2309	0.7875	0.7546	0.7545	0.7295	0.7763	0.7396	0.8117	0.7602
2094	1.2297	0.7870	0.7543	0.7542	0.7293	0.7759	0.7394	0.8111	0.7598
2095	1.2285	0.7865	0.7539	0.7539	0.7291	0.7754	0.7391	0.8105	0.7595
2096	1.2273	0.7860	0.7536	0.7535	0.7290	0.7750	0.7389	0.8099	0.7592
2097	1.2262	0.7854	0.7533	0.7532	0.7288	0.7746	0.7387	0.8094	0.7588

t	$M_{18,t}^{TR}$	$M_{19,t}^{TR}$	$M_{20,t}^{TR}$	$M_{21,t}^{TR}$	$M_{22,t}^{TR}$	$M_{23,t}^{TR}$	$M_{24,t}^{TR}$	$M_{25,t}^{TR}$	$M_{26,t}^{TR}$
2023	7.0045	6.6236	7.6097	7.2081	8.0440	7.8311	8.0001	7.9595	7.6933
2024	1.9453	1.6020	2.4700	2.1093	2.8445	2.6461	2.8043	2.7522	2.5489
2025	1.1727	0.8766	1.6099	1.3001	1.9210	1.7456	1.8878	1.8327	1.6892
2026	0.3293	0.0712	0.6959	0.4272	0.9558	0.7990	0.9280	0.8705	0.7751
2027	0.4458	0.2213	0.7524	0.5200	0.9690	0.8295	0.9463	0.8886	0.8322
2028	0.5392	0.3417	0.7976	0.5944	0.9795	0.8539	0.9607	0.9030	0.8777
2029	0.6139	0.4381	0.8336	0.6540	0.9877	0.8734	0.9720	0.9144	0.9138
2030	0.6735	0.5153	0.8622	0.7017	0.9941	0.8890	0.9806	0.9232	0.9423
2031	0.7211	0.5770	0.8850	0.7397	0.9990	0.9013	0.9872	0.9300	0.9647
2032	0.7591	0.6264	0.9030	0.7701	1.0027	0.9111	0.9921	0.9352	0.9823
2033	0.7893	0.6658	0.9172	0.7943	1.0055	0.9187	0.9957	0.9389	0.9960
2034	0.8134	0.6973	0.9284	0.8135	1.0074	0.9246	0.9981	0.9415	1.0066
2035	0.8324	0.7224	0.9372	0.8288	1.0087	0.9291	0.9996	0.9432	1.0146
2036	0.8475	0.7424	0.9439	0.8408	1.0095	0.9326	1.0005	0.9442	1.0206
2037	0.8593	0.7582	0.9491	0.8503	1.0099	0.9352	1.0007	0.9446	1.0250
2038	0.8687	0.7708	0.9531	0.8577	1.0100	0.9370	1.0005	0.9445	1.0281
2039	0.8759	0.7808	0.9560	0.8635	1.0098	0.9383	0.9999	0.9440	1.0301
2040	0.8815	0.7886	0.9581	0.8680	1.0094	0.9391	0.9991	0.9432	1.0313
2041	0.8859	0.7948	0.9596	0.8714	1.0088	0.9395	0.9980	0.9421	1.0319
2042	0.8891	0.7996	0.9605	0.8740	1.0081	0.9397	0.9966	0.9409	1.0319
2043	0.8915	0.8034	0.9611	0.8758	1.0072	0.9396	0.9952	0.9395	1.0315
2044	0.8933	0.8062	0.9613	0.8772	1.0063	0.9393	0.9936	0.9379	1.0307
2045	0.8945	0.8084	0.9612	0.8781	1.0053	0.9389	0.9919	0.9363	1.0297
2046	0.8953	0.8101	0.9610	0.8787	1.0042	0.9383	0.9901	0.9345	1.0285
2047	0.9013	0.8184	0.9633	0.8835	1.0039	0.9393	0.9894	0.9339	1.0299
2048	0.9004	0.8179	0.9622	0.8827	1.0027	0.9383	0.9873	0.9319	1.0278
2049	0.8994	0.8173	0.9611	0.8819	1.0014	0.9372	0.9852	0.9298	1.0256
2050	0.8985	0.8167	0.9599	0.8811	1.0001	0.9362	0.9831	0.9277	1.0235
2051	0.8975	0.8161	0.9588	0.8803	0.9988	0.9351	0.9810	0.9256	1.0214
2052	0.8966	0.8155	0.9577	0.8794	0.9975	0.9341	0.9789	0.9236	1.0192
2053	0.8956	0.8149	0.9566	0.8786	0.9962	0.9330	0.9768	0.9215	1.0171
2054	0.8947	0.8143	0.9555	0.8778	0.9949	0.9320	0.9747	0.9194	1.0149
2055	0.8938	0.8138	0.9544	0.8770	0.9936	0.9309	0.9727	0.9174	1.0128
2056	0.8928	0.8132	0.9532	0.8762	0.9924	0.9299	0.9706	0.9153	1.0107
2057	0.8919	0.8126	0.9521	0.8754	0.9911	0.9289	0.9685	0.9133	1.0085
2058	0.8910	0.8120	0.9510	0.8746	0.9898	0.9278	0.9664	0.9112	1.0064
2059	0.8901	0.8115	0.9499	0.8738	0.9885	0.9268	0.9643	0.9092	1.0042
2060	0.8891	0.8109	0.9488	0.8730	0.9873	0.9257	0.9623	0.9072	1.0021
2061	0.8882	0.8103	0.9477	0.8722	0.9860	0.9247	0.9602	0.9051	1.0000
2062	0.8873	0.8098	0.9466	0.8714	0.9847	0.9237	0.9581	0.9031	0.9978
2063	0.8864	0.8092	0.9455	0.8706	0.9834	0.9227	0.9560	0.9011	0.9957
2064	0.8855	0.8087	0.9445	0.8698	0.9822	0.9216	0.9540	0.8991	0.9936
2065	0.8846	0.8081	0.9434	0.8691	0.9809	0.9206	0.9519	0.8971	0.9914
2066	0.8837	0.8076	0.9423	0.8683	0.9796	0.9196	0.9499	0.8951	0.9893
2067	0.8828	0.8070	0.9412	0.8675	0.9784	0.9186	0.9478	0.8931	0.9872
2068	0.8819	0.8065	0.9401	0.8667	0.9771	0.9176	0.9458	0.8911	0.9851
2069	0.8810	0.8059	0.9390	0.8660	0.9759	0.9166	0.9437	0.8891	0.9829
2070	0.8801	0.8054	0.9380	0.8652	0.9746	0.9156	0.9417	0.8871	0.9808
2071	0.8792	0.8049	0.9369	0.8644	0.9734	0.9146	0.9396	0.8851	0.9787
2072	0.8784	0.8043	0.9358	0.8637	0.9721	0.9136	0.9376	0.8832	0.9766
2073	0.8775	0.8038	0.9348	0.8629	0.9709	0.9126	0.9356	0.8812	0.9745
2074	0.8766	0.8033	0.9337	0.8621	0.9696	0.9116	0.9335	0.8792	0.9723
2075	0.8757	0.8027	0.9326	0.8614	0.9684	0.9106	0.9315	0.8773	0.9702
2076	0.8749	0.8022	0.9316	0.8606	0.9671	0.9096	0.9295	0.8753	0.9681
2077	0.8740	0.8017	0.9305	0.8599	0.9659	0.9086	0.9275	0.8734	0.9660
2078	0.8731	0.8012	0.9295	0.8591	0.9646	0.9076	0.9255	0.8714	0.9639
2079	0.8723	0.8007	0.9284	0.8584	0.9634	0.9066	0.9234	0.8695	0.9618
2080	0.8714	0.8001	0.9274	0.8576	0.9622	0.9056	0.9214	0.8676	0.9597
2081	0.8706	0.7996	0.9263	0.8569	0.9609	0.9046	0.9194	0.8656	0.9576
2082	0.8697	0.7991	0.9253	0.8562	0.9597	0.9037	0.9174	0.8637	0.9555
2083	0.8689	0.7986	0.9242	0.8554	0.9585	0.9027	0.9154	0.8618	0.9534
2084	0.8680	0.7981	0.9232	0.8547	0.9572	0.9017	0.9135	0.8599	0.9513
2085	0.8672	0.7976	0.9222	0.8540	0.9560	0.9007	0.9115	0.8580	0.9492
2086	0.8664	0.7971	0.9211	0.8532	0.9548	0.8998	0.9095	0.8561	0.9472
2087	0.8655	0.7966	0.9201	0.8525	0.9536	0.8988	0.9075	0.8542	0.9451
2088	0.8647	0.7961	0.9191	0.8518	0.9524	0.8978	0.9055	0.8523	0.9430
2089	0.8639	0.7956	0.9181	0.8511	0.9511	0.8969	0.9036	0.8505	0.9409

t	$M_{18,t}^{TR}$	$M_{19,t}^{TR}$	$M_{20,t}^{TR}$	$M_{21,t}^{TR}$	$M_{22,t}^{TR}$	$M_{23,t}^{TR}$	$M_{24,t}^{TR}$	$M_{25,t}^{TR}$	$M_{26,t}^{TR}$
2090	0.8631	0.7951	0.9170	0.8504	0.9499	0.8959	0.9016	0.8486	0.9389
2091	0.8622	0.7946	0.9160	0.8497	0.9487	0.8950	0.8997	0.8467	0.9368
2092	0.8614	0.7941	0.9150	0.8489	0.9475	0.8940	0.8977	0.8449	0.9347
2093	0.8606	0.7937	0.9140	0.8482	0.9463	0.8931	0.8958	0.8430	0.9327
2094	0.8598	0.7932	0.9130	0.8475	0.9451	0.8921	0.8938	0.8412	0.9306
2095	0.8590	0.7927	0.9120	0.8468	0.9439	0.8912	0.8919	0.8394	0.9286
2096	0.8582	0.7922	0.9110	0.8461	0.9427	0.8902	0.8900	0.8375	0.9265
2097	0.8574	0.7917	0.9100	0.8454	0.9415	0.8893	0.8880	0.8357	0.9245

t	$M_{27,t}^{TR}$	$M_{28,t}^{TR}$	$M_{29,t}^{TR}$	$M_{30,t}^{TR}$	$M_{31,t}^{TR}$	$M_{32,t}^{TR}$	$M_{33,t}^{TR}$	$M_{34,t}^{TR}$	$M_{35,t}^{TR}$
2023	7.8068	7.7698	7.6969	8.9568	8.7604	8.9596	9.0063	8.9639	9.0474
2024	2.6286	2.6104	2.5361	2.8715	2.6998	2.8521	2.8853	2.8227	2.8759
2025	1.7444	1.7401	1.6778	1.9114	1.7849	1.8891	1.9287	1.8592	1.9123
2026	0.8105	0.8171	0.7643	0.9151	0.8246	0.8896	0.9339	0.8579	0.9101
2027	0.8531	0.8677	0.8252	0.9066	0.8486	0.8812	0.9322	0.8529	0.9088
2028	0.8871	0.9080	0.8738	0.8991	0.8672	0.8737	0.9302	0.8480	0.9068
2029	0.9141	0.9398	0.9123	0.8924	0.8814	0.8670	0.9277	0.8433	0.9044
2030	0.9355	0.9649	0.9429	0.8864	0.8919	0.8608	0.9248	0.8386	0.9015
2031	0.9523	0.9847	0.9669	0.8808	0.8995	0.8551	0.9216	0.8340	0.8984
2032	0.9653	1.0000	0.9858	0.8756	0.9048	0.8497	0.9182	0.8294	0.8949
2033	0.9754	1.0119	1.0004	0.8707	0.9082	0.8446	0.9146	0.8248	0.8912
2034	0.9831	1.0210	1.0117	0.8660	0.9100	0.8396	0.9108	0.8203	0.8873
2035	0.9889	1.0278	1.0204	0.8615	0.9107	0.8349	0.9069	0.8158	0.8833
2036	0.9931	1.0329	1.0268	0.8572	0.9104	0.8303	0.9029	0.8114	0.8791
2037	0.9960	1.0364	1.0315	0.8530	0.9093	0.8258	0.8989	0.8069	0.8749
2038	0.9979	1.0389	1.0348	0.8489	0.9076	0.8214	0.8947	0.8025	0.8706
2039	0.9991	1.0404	1.0371	0.8449	0.9055	0.8171	0.8905	0.7981	0.8663
2040	0.9995	1.0411	1.0384	0.8410	0.9029	0.8129	0.8863	0.7937	0.8619
2041	0.9995	1.0413	1.0390	0.8371	0.9001	0.8087	0.8821	0.7894	0.8576
2042	0.9990	1.0409	1.0390	0.8333	0.8970	0.8046	0.8779	0.7850	0.8532
2043	0.9982	1.0402	1.0385	0.8295	0.8937	0.8005	0.8736	0.7807	0.8488
2044	0.9971	1.0392	1.0377	0.8258	0.8902	0.7964	0.8694	0.7764	0.8444
2045	0.9958	1.0379	1.0366	0.8221	0.8867	0.7924	0.8652	0.7722	0.8400
2046	0.9943	1.0365	1.0352	0.8184	0.8830	0.7885	0.8610	0.7679	0.8356
2047	0.9950	1.0374	1.0368	0.8146	0.8806	0.7844	0.8568	0.7637	0.8314
2048	0.9928	1.0352	1.0345	0.8110	0.8766	0.7805	0.8526	0.7595	0.8270
2049	0.9907	1.0329	1.0322	0.8075	0.8726	0.7767	0.8484	0.7554	0.8226
2050	0.9885	1.0306	1.0299	0.8040	0.8686	0.7728	0.8442	0.7512	0.8182
2051	0.9863	1.0284	1.0276	0.8005	0.8647	0.7690	0.8401	0.7471	0.8139
2052	0.9841	1.0261	1.0253	0.7970	0.8607	0.7653	0.8359	0.7430	0.8096
2053	0.9819	1.0238	1.0229	0.7935	0.8568	0.7615	0.8318	0.7389	0.8053
2054	0.9798	1.0216	1.0206	0.7901	0.8529	0.7578	0.8277	0.7349	0.8011
2055	0.9776	1.0193	1.0183	0.7866	0.8490	0.7540	0.8236	0.7308	0.7968
2056	0.9754	1.0170	1.0160	0.7832	0.8451	0.7504	0.8196	0.7268	0.7926
2057	0.9732	1.0148	1.0137	0.7798	0.8412	0.7467	0.8155	0.7229	0.7884
2058	0.9711	1.0125	1.0114	0.7765	0.8374	0.7430	0.8115	0.7189	0.7843
2059	0.9689	1.0102	1.0091	0.7731	0.8336	0.7394	0.8075	0.7150	0.7801
2060	0.9667	1.0079	1.0068	0.7698	0.8298	0.7358	0.8035	0.7111	0.7760
2061	0.9646	1.0057	1.0045	0.7664	0.8260	0.7322	0.7996	0.7072	0.7719
2062	0.9624	1.0034	1.0022	0.7631	0.8222	0.7287	0.7956	0.7033	0.7678
2063	0.9602	1.0011	0.9998	0.7599	0.8185	0.7251	0.7917	0.6995	0.7638
2064	0.9581	0.9989	0.9975	0.7566	0.8148	0.7216	0.7878	0.6957	0.7598
2065	0.9559	0.9966	0.9952	0.7534	0.8111	0.7181	0.7840	0.6919	0.7558
2066	0.9538	0.9944	0.9929	0.7501	0.8074	0.7146	0.7801	0.6882	0.7518
2067	0.9516	0.9921	0.9906	0.7469	0.8037	0.7112	0.7763	0.6845	0.7478
2068	0.9495	0.9898	0.9883	0.7437	0.8001	0.7078	0.7725	0.6808	0.7439
2069	0.9474	0.9876	0.9860	0.7406	0.7965	0.7044	0.7687	0.6771	0.7400
2070	0.9452	0.9853	0.9837	0.7374	0.7929	0.7010	0.7650	0.6734	0.7361
2071	0.9431	0.9831	0.9814	0.7343	0.7893	0.6976	0.7612	0.6698	0.7322
2072	0.9410	0.9808	0.9791	0.7312	0.7857	0.6943	0.7575	0.6662	0.7284
2073	0.9388	0.9785	0.9769	0.7281	0.7822	0.6910	0.7538	0.6626	0.7246
2074	0.9367	0.9763	0.9746	0.7250	0.7786	0.6877	0.7502	0.6591	0.7208
2075	0.9346	0.9740	0.9723	0.7220	0.7751	0.6844	0.7465	0.6556	0.7170
2076	0.9325	0.9718	0.9700	0.7189	0.7717	0.6812	0.7429	0.6521	0.7133
2077	0.9304	0.9695	0.9677	0.7159	0.7682	0.6779	0.7393	0.6486	0.7096
2078	0.9283	0.9673	0.9654	0.7129	0.7647	0.6747	0.7357	0.6451	0.7059
2079	0.9261	0.9650	0.9632	0.7099	0.7613	0.6716	0.7322	0.6417	0.7022
2080	0.9240	0.9628	0.9609	0.7070	0.7579	0.6684	0.7286	0.6383	0.6986
2081	0.9220	0.9606	0.9586	0.7040	0.7545	0.6653	0.7251	0.6349	0.6950
2082	0.9199	0.9583	0.9563	0.7011	0.7512	0.6621	0.7216	0.6316	0.6914
2083	0.9178	0.9561	0.9541	0.6982	0.7478	0.6591	0.7182	0.6283	0.6878
2084	0.9157	0.9539	0.9518	0.6953	0.7445	0.6560	0.7147	0.6250	0.6843
2085	0.9136	0.9516	0.9496	0.6924	0.7412	0.6529	0.7113	0.6217	0.6808
2086	0.9115	0.9494	0.9473	0.6896	0.7380	0.6499	0.7079	0.6185	0.6773
2087	0.9095	0.9472	0.9450	0.6868	0.7347	0.6469	0.7046	0.6152	0.6738
2088	0.9074	0.9450	0.9428	0.6840	0.7315	0.6439	0.7012	0.6120	0.6704
2089	0.9053	0.9427	0.9405	0.6812	0.7282	0.6410	0.6979	0.6089	0.6670

t	$M_{27,t}^{TR}$	$M_{28,t}^{TR}$	$M_{29,t}^{TR}$	$M_{30,t}^{TR}$	$M_{31,t}^{TR}$	$M_{32,t}^{TR}$	$M_{33,t}^{TR}$	$M_{34,t}^{TR}$	$M_{35,t}^{TR}$
2090	0.9033	0.9405	0.9383	0.6784	0.7251	0.6380	0.6946	0.6057	0.6636
2091	0.9012	0.9383	0.9361	0.6756	0.7219	0.6351	0.6913	0.6026	0.6602
2092	0.8992	0.9361	0.9338	0.6729	0.7187	0.6322	0.6880	0.5995	0.6569
2093	0.8971	0.9339	0.9316	0.6702	0.7156	0.6293	0.6848	0.5964	0.6535
2094	0.8951	0.9317	0.9294	0.6675	0.7125	0.6265	0.6816	0.5934	0.6502
2095	0.8931	0.9295	0.9271	0.6648	0.7094	0.6236	0.6784	0.5903	0.6470
2096	0.8910	0.9273	0.9249	0.6621	0.7063	0.6208	0.6752	0.5873	0.6437
2097	0.8890	0.9251	0.9227	0.6595	0.7033	0.6180	0.6721	0.5844	0.6405

t	$M_{36,t}^{TR}$	$M_{37,t}^{TR}$	$M_{38,t}^{TR}$	$M_{39,t}^{TR}$	$M_{40,t}^{TR}$	$M_{41,t}^{TR}$	$M_{42,t}^{TR}$	L_t^{TR}	E_t^{TR}
2023	8.9506	9.1552	4.5715	4.7213	4.6880	5.0545	4.8686	858,000	-6.8924
2024	2.7504	2.8915	1.6461	1.7461	1.6616	1.7846	1.6881	858,000	-6.8976
2025	1.7919	1.9126	0.6263	0.7037	0.6123	0.6569	0.5934	600,000	-7.1229
2026	0.7923	0.8953	0.6348	0.7045	0.6277	0.6787	0.6189	600,000	-7.1278
2027	0.7981	0.8929	0.6409	0.7046	0.6393	0.6953	0.6386	600,000	-7.1328
2028	0.8018	0.8900	0.6451	0.7039	0.6480	0.7079	0.6536	600,000	-7.1377
2029	0.8038	0.8868	0.6479	0.7028	0.6541	0.7171	0.6647	600,000	-7.1426
2030	0.8045	0.8832	0.6493	0.7011	0.6583	0.7236	0.6729	600,000	-7.1476
2031	0.8040	0.8794	0.6498	0.6991	0.6609	0.7280	0.6785	600,000	-7.1526
2032	0.8026	0.8754	0.6495	0.6968	0.6622	0.7307	0.6822	600,000	-7.1574
2033	0.8006	0.8712	0.6486	0.6943	0.6626	0.7320	0.6843	600,000	-7.1622
2034	0.7979	0.8668	0.6471	0.6915	0.6620	0.7322	0.6852	600,000	-7.1669
2035	0.7949	0.8624	0.6452	0.6887	0.6609	0.7316	0.6851	600,000	-7.1716
2036	0.7915	0.8579	0.6430	0.6857	0.6592	0.7303	0.6842	600,000	-7.1761
2037	0.7878	0.8533	0.6406	0.6826	0.6571	0.7284	0.6826	600,000	-7.1806
2038	0.7838	0.8487	0.6379	0.6795	0.6547	0.7262	0.6805	600,000	-7.1850
2039	0.7798	0.8441	0.6351	0.6763	0.6520	0.7235	0.6780	600,000	-7.1893
2040	0.7756	0.8394	0.6321	0.6730	0.6491	0.7207	0.6752	600,000	-7.1935
2041	0.7712	0.8348	0.6291	0.6698	0.6460	0.7175	0.6722	600,000	-7.1977
2042	0.7669	0.8301	0.6259	0.6665	0.6429	0.7143	0.6689	600,000	-7.2016
2043	0.7624	0.8255	0.6227	0.6632	0.6396	0.7109	0.6655	600,000	-7.2055
2044	0.7579	0.8208	0.6195	0.6599	0.6362	0.7074	0.6620	600,000	-7.2093
2045	0.7534	0.8162	0.6162	0.6566	0.6328	0.7038	0.6584	600,000	-7.2129
2046	0.7489	0.8116	0.6130	0.6533	0.6294	0.7002	0.6547	600,000	-7.2164
2047	0.7449	0.8071	0.6102	0.6501	0.6266	0.6974	0.6521	600,000	-7.2199
2048	0.7403	0.8024	0.6068	0.6468	0.6230	0.6935	0.6481	600,000	-7.2232
2049	0.7357	0.7979	0.6034	0.6435	0.6193	0.6896	0.6441	600,000	-7.2265
2050	0.7311	0.7933	0.6000	0.6402	0.6157	0.6857	0.6401	600,000	-7.2298
2051	0.7266	0.7888	0.5967	0.6370	0.6121	0.6819	0.6362	600,000	-7.2330
2052	0.7221	0.7842	0.5933	0.6337	0.6085	0.6781	0.6322	600,000	-7.2363
2053	0.7176	0.7797	0.5900	0.6305	0.6049	0.6743	0.6283	600,000	-7.2396
2054	0.7131	0.7753	0.5867	0.6272	0.6013	0.6705	0.6244	600,000	-7.2429
2055	0.7087	0.7708	0.5834	0.6240	0.5978	0.6667	0.6205	600,000	-7.2462
2056	0.7043	0.7664	0.5802	0.6208	0.5943	0.6629	0.6167	600,000	-7.2497
2057	0.6999	0.7620	0.5769	0.6176	0.5908	0.6592	0.6128	600,000	-7.2532
2058	0.6955	0.7576	0.5737	0.6145	0.5873	0.6554	0.6090	600,000	-7.2569
2059	0.6912	0.7533	0.5705	0.6113	0.5838	0.6517	0.6052	600,000	-7.2607
2060	0.6869	0.7490	0.5673	0.6082	0.5803	0.6480	0.6014	600,000	-7.2645
2061	0.6827	0.7447	0.5641	0.6051	0.5769	0.6443	0.5976	600,000	-7.2685
2062	0.6784	0.7404	0.5610	0.6020	0.5735	0.6407	0.5939	600,000	-7.2726
2063	0.6742	0.7362	0.5578	0.5989	0.5700	0.6370	0.5902	600,000	-7.2768
2064	0.6700	0.7319	0.5547	0.5959	0.5667	0.6334	0.5864	600,000	-7.2811
2065	0.6659	0.7277	0.5516	0.5928	0.5633	0.6298	0.5827	600,000	-7.2855
2066	0.6618	0.7236	0.5485	0.5898	0.5599	0.6262	0.5791	600,000	-7.2899
2067	0.6577	0.7194	0.5455	0.5868	0.5566	0.6226	0.5754	600,000	-7.2944
2068	0.6536	0.7153	0.5424	0.5838	0.5533	0.6190	0.5718	600,000	-7.2989
2069	0.6496	0.7112	0.5394	0.5808	0.5500	0.6155	0.5682	600,000	-7.3034
2070	0.6456	0.7072	0.5364	0.5778	0.5467	0.6119	0.5646	600,000	-7.3079
2071	0.6416	0.7031	0.5334	0.5749	0.5434	0.6084	0.5610	600,000	-7.3124
2072	0.6377	0.6991	0.5305	0.5720	0.5402	0.6049	0.5575	600,000	-7.3168
2073	0.6337	0.6951	0.5275	0.5691	0.5370	0.6015	0.5539	600,000	-7.3213
2074	0.6299	0.6912	0.5246	0.5662	0.5338	0.5980	0.5504	600,000	-7.3257
2075	0.6260	0.6872	0.5217	0.5633	0.5306	0.5946	0.5469	600,000	-7.3300
2076	0.6222	0.6833	0.5188	0.5604	0.5274	0.5911	0.5435	600,000	-7.3343
2077	0.6184	0.6795	0.5159	0.5576	0.5243	0.5877	0.5400	600,000	-7.3385
2078	0.6146	0.6756	0.5131	0.5548	0.5211	0.5843	0.5366	600,000	-7.3427
2079	0.6109	0.6718	0.5102	0.5520	0.5180	0.5810	0.5332	600,000	-7.3468
2080	0.6071	0.6680	0.5074	0.5492	0.5149	0.5776	0.5298	600,000	-7.3509
2081	0.6035	0.6642	0.5046	0.5464	0.5119	0.5743	0.5264	600,000	-7.3550
2082	0.5998	0.6605	0.5018	0.5436	0.5088	0.5710	0.5231	600,000	-7.3590
2083	0.5962	0.6568	0.4991	0.5409	0.5058	0.5677	0.5198	600,000	-7.3631
2084	0.5926	0.6531	0.4963	0.5382	0.5028	0.5644	0.5165	600,000	-7.3671
2085	0.5890	0.6494	0.4936	0.5355	0.4998	0.5612	0.5132	600,000	-7.3711
2086	0.5855	0.6458	0.4909	0.5328	0.4968	0.5579	0.5099	600,000	-7.3751

t	$M_{36,t}^{TR}$	$M_{37,t}^{TR}$	$M_{38,t}^{TR}$	$M_{39,t}^{TR}$	$M_{40,t}^{TR}$	$M_{41,t}^{TR}$	$M_{42,t}^{TR}$	L_t^{TR}	E_t^{TR}
2087	0.5820	0.6421	0.4882	0.5301	0.4938	0.5547	0.5067	600,000	-7.3791
2088	0.5785	0.6386	0.4856	0.5275	0.4909	0.5515	0.5035	600,000	-7.3831
2089	0.5750	0.6350	0.4829	0.5249	0.4880	0.5484	0.5003	600,000	-7.3872
2090	0.5716	0.6315	0.4803	0.5222	0.4851	0.5452	0.4971	600,000	-7.3913
2091	0.5682	0.6280	0.4777	0.5196	0.4822	0.5421	0.4939	600,000	-7.3955
2092	0.5648	0.6245	0.4751	0.5171	0.4794	0.5389	0.4908	600,000	-7.3997
2093	0.5615	0.6210	0.4726	0.5145	0.4765	0.5358	0.4877	600,000	-7.4039
2094	0.5582	0.6176	0.4700	0.5120	0.4737	0.5328	0.4846	600,000	-7.4081
2095	0.5549	0.6142	0.4675	0.5094	0.4709	0.5297	0.4816	600,000	-7.4125
2096	0.5517	0.6108	0.4650	0.5069	0.4681	0.5267	0.4785	600,000	-7.4168
2097	0.5484	0.6075	0.4625	0.5044	0.4654	0.5236	0.4755	600,000	-7.4212

t	S_t^{TR}	O_t^{TR}	U_t^{TR}	I_t^{TR}	R_t^{TR}	W_t^{TR}	DIM_t^{TR}	DIF_t^{TR}	DRM_t^{TR}	DRF_t^{TR}
2023	-3.4494	1,752,000	-3.1136	-2.6592	-0.0098	0.1484	-3.2041	-3.1350	-1.8958	-2.0009
2024	-3.5169	1,752,000	-3.0154	-2.8952	0.0102	1.2022	-2.9845	-2.9089	-1.7666	-1.8803
2025	-3.5785	1,350,000	-3.0227	-2.9188	0.0115	1.6219	-2.9200	-2.8786	-1.8210	-1.9432
2026	-3.6110	1,350,000	-3.0433	-2.9188	0.0116	1.6554	-2.8920	-2.8711	-2.0423	-2.1637
2027	-3.6417	1,350,000	-3.0635	-2.9188	0.0133	1.6384	-2.8658	-2.8482	-2.0770	-2.1951
2028	-3.6708	1,350,000	-3.0666	-2.9188	0.0151	1.5737	-2.9350	-2.9180	-2.1140	-2.2286
2029	-3.6986	1,350,000	-3.0677	-2.9188	0.0179	1.5951	-2.9576	-2.9431	-2.1450	-2.2559
2030	-3.7252	1,350,000	-3.0689	-2.9188	0.0203	1.5675	-2.9730	-2.9539	-2.1760	-2.2845
2031	-3.7507	1,350,000	-3.0700	-2.9188	0.0218	1.5395	-2.9619	-2.9413	-2.2087	-2.3148
2032	-3.7751	1,350,000	-3.0714	-2.9188	0.0223	1.3293	-2.9479	-2.9272	-2.2432	-2.3474
2033	-3.7985	1,350,000	-3.0732	-2.9188	0.0228	1.2430	-2.9480	-2.9274	-2.2354	-2.3421
2034	-3.8209	1,350,000	-3.0749	-2.9188	0.0230	1.2296	-2.9480	-2.9277	-2.2337	-2.3417
2035	-3.8424	1,350,000	-3.0767	-2.9188	0.0230	1.2277	-2.9480	-2.9280	-2.2338	-2.3427
2036	-3.8630	1,350,000	-3.0785	-2.9188	0.0230	1.2259	-2.9480	-2.9283	-2.2335	-2.3418
2037	-3.8828	1,350,000	-3.0807	-2.9188	0.0230	1.2271	-2.9480	-2.9286	-2.2313	-2.3412
2038	-3.9018	1,350,000	-3.0828	-2.9188	0.0230	1.2207	-2.9480	-2.9289	-2.2294	-2.3394
2039	-3.9201	1,350,000	-3.0847	-2.9188	0.0230	1.2226	-2.9480	-2.9291	-2.2287	-2.3394
2040	-3.9376	1,350,000	-3.0862	-2.9188	0.0230	1.2027	-2.9480	-2.9294	-2.2267	-2.3378
2041	-3.9545	1,350,000	-3.0870	-2.9188	0.0230	1.1817	-2.9480	-2.9297	-2.2255	-2.3376
2042	-3.9707	1,350,000	-3.0875	-2.9188	0.0230	1.1615	-2.9480	-2.9300	-2.2243	-2.3378
2043	-3.9864	1,350,000	-3.0875	-2.9188	0.0230	1.1487	-2.9481	-2.9300	-2.2276	-2.3391
2044	-4.0015	1,350,000	-3.0871	-2.9188	0.0230	1.1303	-2.9480	-2.9300	-2.2296	-2.3408
2045	-4.0161	1,350,000	-3.0863	-2.9188	0.0230	1.1132	-2.9481	-2.9300	-2.2313	-2.3430
2046	-4.0301	1,350,000	-3.0849	-2.9188	0.0230	1.0984	-2.9481	-2.9300	-2.2326	-2.3435
2047	-4.0437	1,350,000	-3.0834	-2.9188	0.0230	1.1042	-2.9480	-2.9300	-2.2337	-2.3448
2048	-4.0569	1,350,000	-3.0819	-2.9188	0.0230	1.1008	-2.9481	-2.9300	-2.2341	-2.3456
2049	-4.0695	1,350,000	-3.0803	-2.9188	0.0230	1.1034	-2.9481	-2.9300	-2.2352	-2.3467
2050	-4.0818	1,350,000	-3.0789	-2.9188	0.0230	1.0987	-2.9480	-2.9300	-2.2361	-2.3473
2051	-4.0936	1,350,000	-3.0774	-2.9188	0.0230	1.0989	-2.9481	-2.9300	-2.2368	-2.3481
2052	-4.1049	1,350,000	-3.0760	-2.9188	0.0230	1.0991	-2.9481	-2.9300	-2.2376	-2.3498
2053	-4.1159	1,350,000	-3.0746	-2.9188	0.0230	1.1012	-2.9480	-2.9300	-2.2388	-2.3499
2054	-4.1265	1,350,000	-3.0732	-2.9188	0.0230	1.0953	-2.9481	-2.9300	-2.2398	-2.3505
2055	-4.1367	1,350,000	-3.0719	-2.9188	0.0230	1.0939	-2.9481	-2.9300	-2.2404	-2.3508
2056	-4.1465	1,350,000	-3.0707	-2.9188	0.0230	1.0959	-2.9480	-2.9300	-2.2401	-2.3508
2057	-4.1560	1,350,000	-3.0698	-2.9188	0.0230	1.1025	-2.9481	-2.9300	-2.2402	-2.3511
2058	-4.1651	1,350,000	-3.0690	-2.9188	0.0230	1.1103	-2.9481	-2.9300	-2.2396	-2.3502
2059	-4.1739	1,350,000	-3.0685	-2.9188	0.0230	1.1175	-2.9481	-2.9300	-2.2386	-2.3496
2060	-4.1823	1,350,000	-3.0682	-2.9188	0.0230	1.1229	-2.9481	-2.9300	-2.2384	-2.3495
2061	-4.1904	1,350,000	-3.0681	-2.9188	0.0230	1.1289	-2.9481	-2.9300	-2.2387	-2.3498
2062	-4.1982	1,350,000	-3.0682	-2.9188	0.0230	1.1328	-2.9480	-2.9300	-2.2381	-2.3491
2063	-4.2057	1,350,000	-3.0684	-2.9188	0.0230	1.1347	-2.9481	-2.9300	-2.2381	-2.3489
2064	-4.2129	1,350,000	-3.0688	-2.9188	0.0230	1.1354	-2.9480	-2.9300	-2.2374	-2.3482
2065	-4.2199	1,350,000	-3.0693	-2.9188	0.0230	1.1373	-2.9481	-2.9300	-2.2372	-2.3481
2066	-4.2266	1,350,000	-3.0699	-2.9188	0.0230	1.1368	-2.9481	-2.9300	-2.2362	-2.3468
2067	-4.2330	1,350,000	-3.0706	-2.9188	0.0230	1.1402	-2.9481	-2.9300	-2.2364	-2.3474
2068	-4.2392	1,350,000	-3.0713	-2.9188	0.0230	1.1366	-2.9481	-2.9300	-2.2365	-2.3473
2069	-4.2453	1,350,000	-3.0718	-2.9188	0.0230	1.1382	-2.9481	-2.9300	-2.2367	-2.3468
2070	-4.2511	1,350,000	-3.0723	-2.9188	0.0230	1.1401	-2.9481	-2.9300	-2.2371	-2.3464
2071	-4.2567	1,350,000	-3.0726	-2.9188	0.0230	1.1376	-2.9481	-2.9300	-2.2375	-2.3467
2072	-4.2622	1,350,000	-3.0728	-2.9188	0.0230	1.1333	-2.9481	-2.9300	-2.2373	-2.3466
2073	-4.2675	1,350,000	-3.0729	-2.9188	0.0230	1.1312	-2.9481	-2.9300	-2.2372	-2.3472
2074	-4.2727	1,350,000	-3.0730	-2.9188	0.0230	1.1344	-2.9481	-2.9300	-2.2368	-2.3454
2075	-4.2778	1,350,000	-3.0730	-2.9188	0.0230	1.1356	-2.9481	-2.9300	-2.2367	-2.3462
2076	-4.2827	1,350,000	-3.0729	-2.9188	0.0230	1.1382	-2.9480	-2.9300	-2.2367	-2.3453
2077	-4.2874	1,350,000	-3.0727	-2.9188	0.0230	1.1394	-2.9481	-2.9300	-2.2366	-2.3456
2078	-4.2921	1,350,000	-3.0724	-2.9188	0.0230	1.1375	-2.9481	-2.9300	-2.2370	-2.3461
2079	-4.2966	1,350,000	-3.0720	-2.9188	0.0230	1.1348	-2.9481	-2.9300	-2.2376	-2.3459
2080	-4.3009	1,350,000	-3.0716	-2.9188	0.0230	1.1304	-2.9481	-2.9300	-2.2377	-2.3462
2081	-4.3052	1,350,000	-3.0712	-2.9188	0.0230	1.1326	-2.9481	-2.9300	-2.2373	-2.3462
2082	-4.3093	1,350,000	-3.0708	-2.9188	0.0230	1.1319	-2.9481	-2.9300	-2.2374	-2.3463
2083	-4.3133	1,350,000	-3.0705	-2.9188	0.0230	1.1322	-2.9481	-2.9300	-2.2370	-2.3459
2084	-4.3172	1,350,000	-3.0702	-2.9188	0.0230	1.1332	-2.9480	-2.9300	-2.2373	-2.3458
2085	-4.3209	1,350,000	-3.0699	-2.9188	0.0230	1.1311	-2.9481	-2.9300	-2.2369	-2.3461
2086	-4.3244	1,350,000	-3.0698	-2.9188	0.0230	1.1342	-2.9481	-2.9300	-2.2368	-2.3456
2087	-4.3279	1,350,000	-3.0697	-2.9188	0.0230	1.1295	-2.9481	-2.9300	-2.2376	-2.3457
2088	-4.3312	1,350,000	-3.0696	-2.9188	0.0230	1.1276	-2.9481	-2.9300	-2.2379	-2.3459
2089	-4.3344	1,350,000	-3.0697	-2.9188	0.0230	1.1250	-2.9481	-2.9300	-2.2377	-2.3467

t	S_t^{TR}	O_t^{TR}	U_t^{TR}	I_t^{TR}	R_t^{TR}	W_t^{TR}	DIM_t^{TR}	DIF_t^{TR}	DRM_t^{TR}	DRF_t^{TR}
2090	-4.3374	1,350,000	-3.0698	-2.9188	0.0230	1.1233	-2.9481	-2.9300	-2.2378	-2.3469
2091	-4.3403	1,350,000	-3.0700	-2.9188	0.0230	1.1214	-2.9481	-2.9300	-2.2382	-2.3470
2092	-4.3431	1,350,000	-3.0703	-2.9188	0.0230	1.1213	-2.9481	-2.9300	-2.2387	-2.3470
2093	-4.3457	1,350,000	-3.0706	-2.9188	0.0230	1.1212	-2.9481	-2.9300	-2.2385	-2.3473
2094	-4.3483	1,350,000	-3.0709	-2.9188	0.0230	1.1210	-2.9480	-2.9300	-2.2388	-2.3474
2095	-4.3507	1,350,000	-3.0712	-2.9188	0.0230	1.1205	-2.9481	-2.9300	-2.2389	-2.3472
2096	-4.3530	1,350,000	-3.0715	-2.9188	0.0230	1.1207	-2.9481	-2.9300	-2.2392	-2.3473
2097	-4.3552	1,350,000	-3.0719	-2.9188	0.0230	1.1213	-2.9481	-2.9300	-2.2395	-2.3475

3. Bounding Results

The program also bounds certain assumptions for reasonableness. If a bound is breached in a particular year, then the final value is set to that bound and the program continues running. In both the runs without parameter uncertainty and with parameter uncertainty for the estimated mean, the bounds are:

1. Total Fertility Rate: The total fertility rate is bounded by 0 on the low end. We set a high bound of 100,000, although the highest value reached is 4.62.
2. LPR New Arrival Immigration: 0 persons through 30 million persons.
3. Other-than-LPR Immigration: 100,000 persons through 15 million persons.
4. Economic VAR Framework (Unemployment Rate, Inflation Rate, and Real Interest Rate): In cases where the nominal interest rate is less than 0 when combining the inflation rate and real interest rate, the real interest rate is set such that the nominal interest rate is 0. Bounds of plus and minus 40 percent were put on the inflation rate and real interest rate. For the real interest rate, the plus/minus 40-percent bound is imposed before the bound requiring a nominal interest rate of at least 0 percent.
5. Percentage Change in Average Real Wage: The percentage change in average real wage is bounded by plus and minus 40 percent.

Table V.4 shows the percentage of runs where a bound was applied in at least one year.

Table V.4 – Percentage of Runs Where a Bound Was Applied in at Least One Year

Assumption	Without Parameter Uncertainty	With Parameter Uncertainty for the Estimated Mean
Total Fertility Rate	0.3	0.5
LPR New Arrival Immigration	0.0	0.1
Other-than-LPR Immigration	0.2	1.2
Inflation Rate	0.0	0.0
Real Interest Rate	53.7	53.2
Real Average Covered Wage	0.0	0.0

All instances of applying a bound to the real interest rate are caused by the resulting nominal interest rate being less than zero. Note that although many simulations have at least one year where the real interest rate is restricted by the bound, the bound is typically not breached in many years in the same simulation. Without parameter uncertainty, for each run with at least one year restricted by the bound, the average number of years that were restricted was about four per run. With parameter uncertainty for the expected mean, this value is about nine per run. These frequencies are consistent with the observation that the zero lower bound on nominal interest rates occasionally constrains monetary policy.

The other assumptions (those not listed here) do not have any bounds imposed, because the transformations of the original data make bounding unnecessary. The regressions for the transfer rate, legal emigration rate, unemployment rate, male and female disability incidence rates, and male and female disability recovery rates are all performed on data transformed to be between 0 and 100 percent.

Although it is not a bounding issue per se, three runs with parameter uncertainty for the estimated mean produced a negative number of LPR immigrants plus citizens for at least one year/sex/age combination. When this occurred, the number of LPR immigrants plus citizens for these particular year/sex/age combinations was reset to one.

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